© Revista de Matemática: Teoría y Aplicaciones 2021 **28**(1) : 39–53
cimpa – ucr issn: 1409-2433 (Print), 2215-3373 (Online)
doi: https://doi.org/10.15517/rmta.v28i1.33773

EXISTENCE CONDITIONS FOR k-BARYCENTRIC OLSON CONSTANT

CONDICIONES DE EXISTENCIA PARA LA CONSTANTE DE OLSON k-BARICÉNTRICA

Luz Marchan[*] Oscar Ordaz[†] José Salazar[‡]
Felicia Villarroel[§]

Received: 16/Jul/2018; Revised: 25/Jun/2020;
Accepted: 30/Oct/2020

[*]Escuela Superior Politécnica del Litoral, Departamento de Matemáticas, Facultad de Ciencias Naturales, Guayaquil, Ecuador. E-Mail: luzelimarchan@gmail.com

[†]Universidad Central de Venezuela, Departamento de Matemáticas y Laboratorio LaTecS, Centro ISYS, Facultad de Ciencias, Caracas, Venezuela. E-Mail: oscarordaz55@gmail.com

[‡]Universidad de Oriente, Departamento de Matemáticas, Núcleo Sucre, Cumaná, Venezuela. E-Mail: jsalazar@udo.edu.ve

[§]Universidad de Oriente, Cumaná, Departamento de Matemáticas, Núcleo Sucre, Venezuela. E-Mail: feliciavillarroel@gmail.com

Abstract

Let $(G,+)$ be a finite abelian group and $3 \leq k \leq |G|$ a positive integer. The k-barycentric Olson constant denoted by $BO(k,G)$ is defined as the smallest integer ℓ such that each set A of G with $|A| = \ell$ contains a subset with k elements $\{a_1,\dots,a_k\}$ satisfying $a_1 + \cdots + a_k = ka_j$ for some $1 \leq j \leq k$. We establish some general conditions on G assuring the existence of $BO(k,G)$ for each $3 \leq k \leq |G|$. In particular, from our results we can derive the existence conditions for cyclic groups and for elementary p-groups $p \geq 3$. We give a special treatment over the existence condition for the elementary 2-groups.

Keywords: finite abelian group; zero-sum problem; baricentric-sum problem; Davenport constant; k-barycentric Olson constant.

Resumen

Sean $(G,+)$ un grupo abeliano finito y $3 \leq k \leq |G|$ un entero positivo. La constante de Olson k-baricéntrica, denotada por $BO(k,G)$, se define como el menor entero positivo ℓ tal que todo conjunto A de G con $|A| = \ell$ contiene un subconjunto con k elementos $\{a_1,\dots,a_k\}$ que satisface $a_1+\cdots+a_k = ka_j$ para algún $1 \leq j \leq k$. Establecemos algunas condiciones generales sobre G asegurando la existencia de $BO(k,G)$ para cada $3 \leq k \leq |G|$. En particular, a partir de nuestros resultados podemos determinar las condiciones de existencia para los grupos cíclicos y para los p-grupos elementales con $p \geq 3$. Damos un tratamiento especial a la condición de existencia para los 2-grupos elementales.

Palabras clave: grupos abelianos finitos; problemas de suma-cero; problemas de suma baricéntricas; constante de Davenport; constante k-baricéntrica de Olson.

Mathematics Subject Classification: 11B30, 11B75, 94B05, 94B65, 51E22.

1 Introduction

We recall some standard terminology and notation. We denote by $\mathbb{N}$ the positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For abelian groups, we use additive notation and we denote the neutral element by 0. For $n \in \mathbb{N}$, let C_n denotes a cyclic group of order n. For each finite abelian group there exists $1 < n_1 \mid \cdots \mid n_r$ such that $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$. The integer n_r is called the exponent of G, denoted $\exp(G)$. The integer r is called the rank of G, denoted $r(G)$. For a prime p, the p-rank of G, denoted $r_p(G)$, is the smallest number i such that n_i is divisible by p. For a prime number p we denote by $\mathbb{F}_p$ the field with p elements.

We say that G is a p-group if its exponent is a prime power and we say that G is an elementary p-group if the exponent is a prime (except for the trivial group). Let G be an abelian finite group. The sumset of two subsets A and B of G will be denoted by $A + B = \{a + b : a \in A \ \wedge \ b \in B\}$. We denote the sum of the elements of a subset S of G by $\sigma(S)$. Furthermore, for an integer k, let $\sum_k(A) = \{\sigma(B) : B \subseteq A \ \wedge \ |B| = k\}$. Finally, for t an integer, we denote by $t \cdot A$ the set of multiples $t \cdot A = \{ta : a \in A\}$.

For a finite abelian group $(G, +)$ and $3 \leq k \leq |G|$ a positive integer, the k-barycentric Olson constant denoted by $BO(k, G)$ is the smallest ℓ such that each set A with $|A| = \ell$ over G has a subset with k elements $\{a_1, \ldots, a_k\}$ satisfying $a_1 + \cdots + a_k = ka_j$ for some $1 \leq j \leq k$. This set with k elements is called a k-barycentric set and a_j is called its barycenter. Notice that a k-barycentric set can be written as a weighted zero-sum set that is:

$$a_1 + \cdots + (1 - k)a_j + \cdots + a_k = 0.$$

So that the k-barycentric Olson constant can be seen as a classical example of a weighted zero-sum constant over a finite abelian group. This constant together with related invariants have been studied in the literature [5, 6]. The aim of the present work is to establish conditions on G for the existence of $BO(k, G) \leq |G|$ for each $3 \leq k \leq |G|$. That is to say, for each $3 \leq k \leq |G|$ there exists a k-barycentric set.

Existence conditions of the k-barycentric Olson constant with $3 \leq k \leq |G|$ were initially considered in [14] with the study on cyclic groups using the Orbits Theory. In [13] Ordaz, Plagne and Schmid researched on the existence conditions of $BO(k, G)$ with $|G| - 2 \leq k \leq |G|$ over finite abelian groups G in general; their results were Lemma 1 and Proposition 1. In case there are no k-barycentric sets in G we write $BO(k, G) = |G| + 1$.

Lemma 1 *([13] , Lemma 3.1) Let G be a finite abelian group. Then*

$$\sigma(G) = \begin{cases} b^* & \text{if } r_2(G) = 1 \text{ and } b^* \text{ denote the only element with order } 2, \\ 0 & \text{in other case.} \end{cases}$$

Hence we have that:

$$BO(|G|, G) = \begin{cases} |G| + 1 & \text{if } r_2(G) = 1, \\ |G| & \text{in other case.} \end{cases}$$

The following result gives the values of $BO(|G|-1, G)$ and $BO(|G|-2, G)$.

Proposition 1 *([13] , Proposition 3.2) Let G be a finite abelian group. Then for $|G| \geq 2$, we have:*

$$BO(|G|-1, G) = \begin{cases} |G|-1 & \textit{if } r_2(G) = 1, \\ |G|+1 & \textit{in other case.} \end{cases}$$

and for $|G| \geq 3$, we have:

$$BO(|G|-2, G) = \begin{cases} |G|-2 & \textit{if } |G| \textit{ is odd,} \\ |G|+1 & \textit{if } \exp(G) = 2 \textit{ or } |G| = 4, \\ |G|-1 & \textit{in other case.} \end{cases}$$

In the Lemma 1 is determine the conditions of existence of $BO(|G|, G)$ and in the Proposition 1 is determine the conditions of existence of $BO(k, G)$ with $|G|-2 \leq k \leq |G|-1$.

In the same order of ideas of the above results, the main goal of our paper is to show that the finite abelian groups G with $r_2(G) = 0$ and the finite abelian groups G with $r_2(G) = 1$ contain a k-barycentric set for each $3 \leq k \leq |G|-3$. Notice that the cyclic groups C_n are members of these groups since $r_2(C_n) = 0$ if and only if n is odd and $r_2(C_n) = 1$ if and only if n is even. Similarly, elementary p-groups with $p \neq 2$, are members of the above groups since $r_2(C_p^m) = 0$. In consequence our results solve completely the existence conditions of the k-barycetric Olson constant, for cyclic groups and for elementary p-groups. It is clear that the elementary 2-groups are outside the above groups and then we have a special consideration for its existence conditions for $BO(k, C_2^m)$. As a second goal in our investigation, for some G and k, we give an exact value for $BO(k, G)$ when it exists. For example, we show that $BO(|G|-3, G) = |G|-2$ for the abelian groups G with $r_2(G) = 1$, $|G| \geq 8$ and non multiple of 3. Moreover, we show that $BO(3^m - 3, C_3^m) = 3^m - 2$, in this case $r_2(C_3^m) = 0$.

The organization of the paper besides this introduction and the conclusion, is as follows: a first section on preliminaries, a second section on existence conditions for general finite abelian groups and finally, a third section on some existence conditions for elementary 2-groups.

2 Preliminaries

In this section we give some previous and useful results.

Remark 1 *Let G be a finite abelian group. Then*

i. $r_2(G) = 0$ *if and only if* $|G|$ *is odd.*

ii. $r_2(G) = 1$ *implies that* $|G|$ *is even. Let* $b^* \in G$ *be the only element of order* 2*. It is clear that for cyclic groups we have the equivalence* $r_2(C_n) = 1$ *if and only if* n *is even. Also we have that* $r_2(C_p^m) = 0$ *for* $p \neq 2$*. Moreover, if* $t = r_2(G) \geq 1$*, then* $|G|$ *is even and* G *has* $2^t - 1$ *elements of order* 2*.*

Proposition 2 *Let G be a finite abelian group with $|G| \geq 8$ such that $r_2(G) = 1$ and $3 \nmid |G|$. Then.*

i. $-3 \cdot G = G$*.*

ii. Let $a \in G$ *and* $S_a = \{x \in G : 2x = a\}$*. Then* $|S_a| \leq 2$*.*

Proof. i. Let $\phi : G \to -3 \cdot G$ be given by $\phi(a) = -3a$ where $-3 \cdot G = \{3(-a) : a \in G\}$. Let $y = 3(-a) \in G$, then exits $a \in G$ such that $\phi(a) = -3a = 3(-a) = y$, therefore ϕ is surjective. Assuming that $\phi(a_1) = \phi(a_2)$, then $-3a_1 = -3a_2$, so that, $3(a_1 - a_2) = 0$. Since $3 \nmid |G|$, then $a_1 = a_2$, i.e., ϕ is injective. Then $|G| = |-3 \cdot G|$. Since $-3 \cdot G \subseteq G$ and G is finite, then $-3 \cdot G = G$.

ii. Assuming we have three different elements $a_1, a_2, a_3 \in S_a$, then $2a_1 = 2a_2$ and $2a_1 = 2a_3$, in consequence $2(a_1 - a_2) = 0$ and $2(a_1 - a_3) = 0$.

Since a_1, a_2, a_3 are different, then $a_1 - a_2 = b^*$ and $a_1 - a_3 = b^*$, where b^* is the only element of order 2 in G. Hence $a_2 = a_3$, contradiction. So that $|S_g| \leq 2$. ■

We have the following result:

Proposition 3 *If $m \geq 2$, then $3^m - 2 \leq BO(3^m - 3, C_3^m)$.*

Proof. Let $A = C_3^m \setminus \{-a, -b, 0\}$ be a $(3^m - 3)$-subset over C_3^m with $a + b \neq 0$. Since $\sigma(C_3^m) = 0$ and $\sigma(C_3^m) = \sigma(A) + \sigma(A^c)$ where $A^c = \{-a, -b, 0\}$, then $\sigma(A) = -\sigma(A^c) \Rightarrow \sigma(A) = -(-a - b + 0) \Rightarrow \sigma(A) = a + b \neq 0$. Moreover we have that $(3^m - 3)a = (3^m - 3)(3a) = (3^{m-1} - 1)0 = 0$ for all $a \in A \subset C_3^m$ since $3x = 0$ for all $x \in C_3^m$. Therefore, there exists a $(3^m - 3)$-subset A over C_3^m such that $\sigma(A) \neq (3^m - 3)a$ for all $a \in A$, i.e., $3^m - 2 \leq BO(3^m - 3, C_3^m)$. ■

We need the following result:

Proposition 4 *Let A be a k-subset of C_2^m such that $3 \leq k \leq 2^m$.*

i. *If k is even, then A is a k-barycentric set if and only if $\sigma(A) = 0$.*

ii. *If k is odd, then A is a k-barycentric set if and only if $\sigma(A) \in A$.*

iii. *Let $A^c = C_2^m \setminus A$ the complement of A, then $|A| = 2^m - |A^c|$.*

iv. $\sigma(A) = \sigma(A^c)$.

v. $0 \notin \sum_2 C_2^m$.

Proof. It follows directly. ■

The following lemma guarantees the existence of k-sets of zero-sum with $4 \leq k \leq \frac{|G|}{2} - 1$ in a finite abelian group.

Lemma 2 *([3] , Lemma 7.1) Let G be a finite abelian group de orden $|G| \geq 2$.*

1. *There exists a squarefree zero sequence $S \in F(G)$ with $|S| = |G| - 1$.*
2. *Let $0 \neq g_0 \in G$ and $1 \leq k \leq \frac{|G|}{2} - 1$ with $k \neq 2$, if G is an elementary 2-group. Then there exist a squarefree zero sequence $S \in F(G)$ with $g_0 \nmid S$ and $|S| = k$.*

The following corollary is a consequence of the above lemma.

Corollary 1 *Let G is an elementary 2-group de orden $|G| \geq 3$ such that $0 \neq x \in G$ and $4 \leq k \leq \frac{|G|}{2} - 1$. Then there exist a k-set A of zero-sum in G such that $x \notin A$.*

3 Existence conditions of $BO(k, G)$ for general abelian groups

Let G be a finite. In the following two theorems, the values $r_2(G) = 0$ or $r_2(G) = 1$ are considered to give an existence condition in the order G to have a k-barycentric set, for each $3 \leq k \leq |G| - 3$. Notice that from Remark 1 the parity of $|G|$ is used and depends on $r_2(G) = 0$ or $r_2(G) = 1$. Observe that the fact $r_2(G) = 0$ means that for each $g \in G$ we have $-g \neq g$. The results provided in this section allow us to establish the existence of $BO(k, G)$ with $3 \leq k \leq |G| - 3$ for cyclic groups and elementary p-groups. A relationship between the Harborth $g(G)$ and the k-barycentric Olson $BO(k, G)$ constants is presented. From these relations, we give exact values of $BO(k, G)$ for some groups where $g(G)$ exists. Finally we identify some conditions on certain groups G in order to provide the exact values of $BO(|G| - 3, G)$.

Theorem 1 *Let G be a finite abelian group such that $r_2(G) = 0$ and $3 \leq k \leq |G| - 3$. Then $BO(k, G) \leq |G|$.*

Proof. Assuming $|G| \geq 9$. Let A be a zero-sum set of G such that $|A| = 3$ with $0 \notin A$ and we consider $B = \{-a : a \in A\}$. Notice that the sets $A \cup \{0\}$, $A \setminus \{a\} \cup B \setminus \{-a\}\} \cup \{0\}$ for some $a \in A$ and $A \cup B \cup \{0\}$ over G are k-barycentric, then $BO(k, G) \leq |G|$ for $k = 4$, 5 y 7.

Let $C = G \setminus (A \cup B \cup \{0\})$. Notice that since $|G| \geq 9$ and also odd then $|C| \geq 2$ is even. Moreover for all $c \in C$ we can see that $-c \in C$, assuming the contrary, we have a contradiction. Hence there exists $E \subseteq C$ with $2 \leq |E| \leq |C|$ conformed by elements a and its opposite. Since $|E|$ is even then $E \cup A \cup \{0\}$ or $E \cup A \cup B \cup \{0\}$ constitute the k-barycentric sets even or odd with barycenter 0, over G. Notice that $6 \leq k \leq |G| - 3$ with $k \neq 7$.

Moreover, since for all $0 \neq g \in G$ the set $\{g, -g, 0\}$ over G is a zero-sum then $BO(3, G) \leq |G|$.

Now, we consider the finite abelian groups G of order 3, 5 and 7. Observe that these groups are cyclic. In what follows we consider the existence of $BO(k, G)$. By Lemma 1 we have that $BO(3, C_3) = 3$, $BO(5, C_5) = 5$ and $BO(7, C_7) = 7$. Moreover by Proposition 1 we have that $BO(4, C_5)$ and $BO(6, C_7)$ does not exist and $BO(3, C_5) = 3$ and $BO(5, C_7) = 5$. Moreover, the 4-subset $A = \{0, 1, 2, 4\}$ over C_7 a zero-sum and $0 \in A$, in consequence $BO(4, C_7) \leq 7$ and for all $0 \neq a \in C_7$ the 3-subset $A = \{0, a, -a\}$ a zero-sum and $0 \in A$, hence $BO(3, C_7) \leq 7$. ■

The following two corollaries are a direct consequence of the above theorem.

Corollary 2 *Let C_n be a cyclic group such that $r_2(C_n) = 0$ and $3 \leq k \leq n - 3$. Then $BO(k, C_n) \leq n$.*

Corollary 3 *Let C_p^m be a elementary p-group such that $r_2(C_p^m) = 0$ and $3 \leq k \leq p^m - 3$. Then $BO(k, C_p^m) \leq p^m$.*

Theorem 2 *Let G be a finite abelian group such that $r_2(G) = 1$ and $3 \leq k \leq |G| - 3$ a positive integer. Then $BO(k, G) \leq |G|$.*

Proof. Assuming $|G| \geq 8$. Let $b^* \in G$ the only element of order 2. Let A be a 3-subset with zero-sum over G such that $b^* \in A$, $0 \notin A$ and $B = \{-a : a \in A\} \setminus \{b^*\}$. It is clear that the sets $A \cup \{0\}$ and $A \setminus \{b^*\} \cup B \cup \{0\}$ over G are barycentric with barycenter 0. Hence $BO(k, G) \leq |G|$ for $k = 4$ and 5.

Consider now, the set $C = G \setminus (A \cup B \cup \{0\})$. By Remark 1 G is even and then since $|A \cup B \cup \{0\}| = 6$ we have that $|C| \geq 2$ is even. Moreover for each $c \in C$ we have $-c \in C$, assuming the contrary we have a contradiction. Therefore there exists $E \subseteq C$ with zero-sum and $2 \leq |E| \leq |C|$ conformed by elements in C and its opposite. Hence the sets $E \cup A \cup \{0\}$ and $E \cup A \setminus \{b^*\} \cup B \cup \{0\}$ give the k-barycentric sets over G, k even and odd with barycenter 0 such that $6 \leq k \leq |G| - 3$.

Moreover, since for all $b^* \neq g \in G$ the set $\{g, -g, 0\}$ of G has zero-sum then $BO(3, G) \leq |G|$.

Now, we consider the finite abelian groups G of order 4 and 6. Observe that these groups are cyclic. In what follows we consider the existence of $BO(k, G)$. By Lemma 1 we have that $BO(4, C_4)$ and $BO(6, C_6)$ does not exist. Moreover by Proposition 1 we have that $BO(3, C_4) = 3$ and $BO(5, C_6) = 5$. Moreover, for all $3 \neq a \in C_6$ the 3-subset $A = \{0, a, -a\}$ a zero-sum and $0 \in A$, hence $BO(3, C_6) \leq 6$. ■

The following corollary is a consequence of the above theorem.

Corollary 4 *Let C_n be a cyclic group such that $r_2(C_n) = 1$ and $3 \leq k \leq n-3$. Then $BO(k, C_n) \leq n$.*

Theorem 3 *Let G be a finite abelian group with $|G| \geq 8$, $r_2(G) = 1$ and $3 \nmid |G|$. Then $BO\,(|G| - 3, G) = |G| - 2$.*

Proof. Let $b^* \in G$ be the only element with order 2. Let $A \subseteq G$ be such that $|A| = |G| - 2$. Assuming that $A = G \setminus \{a_1, a_2\}$ and consider $B = A \setminus [\{b^* + 2a_1 - a_2, b^* + 2a_2 - a_1\} \cup S_{a_1+a_2-b^*}]$. Since $|G| \geq 8$ then $|B| = |A| - 2 - |S_{a_1+a_2-b^*}| \geq (|G| - 2) - 2 - 2 = |G| - 6 > 0$. Hence $B \neq \emptyset$.

Let $b \in B \subseteq A$ be and consider the $(|G| - 3)$-subset $A \setminus \{b\}$ of A and we will see that $A \setminus \{b\}$ is a $(|G| - 3)$-barycentric set of A. We have that $\sigma(A \setminus \{b\}) = \sigma(A) - \sigma(b) = \sigma(G) - a_1 - a_2 - b = b^* - a_1 - a_2 - b$. Moreover, by Proposition 2 we have $-3 \cdot G = G$, then $\sigma(A \setminus \{b\}) = b^* - a_1 - a_2 - b = -3c$ for some $c \in G$. If $c = a_1$, then $b = b^* + 2a_1 - a_2 \notin B$, contradiction. If $c = a_2$, then $b = b^* + 2a_2 - a_1 \notin B$, contradiction. if $c = b$, then $2b = a_1 + a_2 - b^*$, in consequence $b \in S_{a_1+a_2-b^*} \nsubseteq B$, contradiction. Hence, $c \in G \setminus \{a_1, a_2, b\} = A \setminus \{b\}$ and therefore $\sigma(A \setminus \{b\}) = b^* - a_1 - a_2 - b = -3c = (|G| - 3)\, c$, for some $c \in A \setminus \{b\}$. Hence $A \setminus \{b\}$ is a $(|G|-3)$-barycentric set of A, i.e., $BO\,(|G|-3, G) \leq |G|-2$.

Now we see, $|G|-2 \leq BO\,(|G|-3, G)$. Consider the set $B = G \setminus [\{0, b^*\} \cup S_{b^*}]$. Since $|G| \geq 8$ then, $|B| = |G|-2-|S_{b^*}| \geq |G|-2-2 = |G|-4 > 0$. So that $B \neq \emptyset$.

Let $b \in B$ be, then $2b \neq b$ and $2b \neq b^*$ since if $2b = b^*$,$b \in S_{b^*}$. Consider $A = G \setminus \{b^*, b, 2b\}$, then $|A| = |G|-3$ and $\sigma(A) = \sigma(G \setminus \{b^*, b, 2b\}) = \sigma(G) - b^* - b - 2b = b^* - b^* - b - 2b = -3b$. If $\sigma(A) = -3c$ for some $c \in A$, then $-3b = -3c$, in consequence $b = c$, this is a contradiction with the fact that $b \notin A$, that is to say, A it is not a $(|G|-3)$-barycentric set of G. So that $|G|-2 \leq BO\,(|G|-3, G)$. Therefore, $BO\,(|G|-3, G) = |G|-2$. ■

The following corollary is a consequence of the above theorem.

Corollary 5 *Let C_n be a cyclic group with $n \geq 8$, $r_2(C_n) = 1$ and $3 \nmid n$. Then $BO\,(n-3, G) = n-2$.*

Theorem 4 *Let $m \geq 2$ be then we have that $BO(3^m-3, C_3^m) = 3^m-2$.*

Proof. By Proposition 3 we have that $3^m-2 \leq BO(3^m-3, C_3^m)$. Let A be a $(k-2)$-subset over C_3^m. If $\sigma(A) \in A$, then the (3^m-3)-subset $B = A \setminus \{\sigma(A)\}$ of A is a zero-sum. So that $\sigma(B) = 0 = (3^m-3)b$ for each $b \in B$. Hence $B = A \setminus \{\sigma(A)\}$ is a (3^m-3)-barycentric set.

Assuming that $\sigma(A) \notin A$, then $\sigma(A) \in A^c$ where A^c is a 2-subset over C_3^m. In consequence $A^c = \{\sigma(A), a\}$ with $\sigma(A) \neq a$. Since $\sigma(C_3^m) = 0$ and $\sigma(C_3^m) = \sigma(A) + \sigma(A^c)$, then $\sigma(A^c) = -\sigma(A) \Rightarrow \sigma(A) + a = -\sigma(A) \Rightarrow a + 2\sigma(A) = 0 = 3a \Rightarrow a = \sigma(A)$, a contradiction with the fact that $\sigma(A) \neq a$. Therefore, $BO(3^m-3, C_3^m) = 3^m-2$. ■

In what follows we consider the Harborth constant and we give its relationship with the k-barycentric Olson constant.

Definition 1 *Let G be a finite abelian group. The Harborth constant, denoted $g(G)$, is defined as the smallest positive integer ℓ such that each set $A \subseteq G$ with $|A| = \ell$ contains a subset B with $|B| = \exp(G)$ with zero-sum.*

The following remark and theorem establishes a relationship between the Harborth constant and the zero-sum problem.

Remark 2 *Kemnitz showed $g(C_p^2) = 2p-1$ for $p \in \{3, 5, 7\}$ in [9]. In particular, $g(C_3^2) = 5$. More recently Gao and Thangadurai [4] showed $g(C_p^2) = 2p-1$ for prime $p \geq 67$ and $g(C_4^2) = 9$. In [2] we can find other values for elementary 3-group; for example $g(C_3^3) = 10$, $g(C_3^3) = 21$, $g(C_3^5) = 46$ and $112 \leq g(C_3^6) \leq 114$ [1, 7, 8, 12].*

Theorem 5 *([11], Theorem 1.1)*

$$g(C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \text{if } n \text{ is even,} \\ 2n+3 & \text{if } n \text{ is odd.} \end{cases}$$

The following result determines the exact values of $BO(\exp(G), G)$ in finite abelian groups G where $g(G)$ there exists.

Theorem 6 *Let G be a finite abelian group where $g(G)$ exists. Then $BO(\exp(G), G) = g(G)$.*

Proof. Let $A \subseteq G$ be such that $|A| = g(G)$, then there exits $B \subseteq A$ with $|B| = \exp(G)$ such that $\sigma(B) = 0$. Therefore $\sigma(B) = 0 = \exp(G)b$ for all $b \in B$. Hence B is a $(\exp(G))$-barycentric subset of A, so that $BO(\exp(G), G) \leq g(G)$. Assuming that $A \subseteq G$ with $|A| = BO(\exp(G), G)$, then A contains a $(\exp(G))$-subset such that $\sigma(B) = (\exp(G))b = 0$ for all $b \in B$. So that B is a $(\exp(G))$-subset with zero-sum of A, that is to say $g(G) \leq BO(\exp(G), G)$. Therefore, $BO(\exp(G), G) = g(G)$. ■

The following result gives the exact values of $BO(\exp(G)+1, G)$ for finite abelian groups where $g(G)$ exists and $g(G) \geq \exp(G)+1$.

Theorem 7 *Let G be a finite abelian group such that $g(G)$ exists and $g(G) \geq \exp(G)+1$. Then $BO(\exp(G)+1, G) = g(G)$.*

Proof. Let $A \subseteq G$ be such that $|A| = g(G) \geq \exp(G)+1$, then there exists $B \subseteq A$ with $|B| = \exp(G)$ such that $\sigma(B) = 0$. Now, since $|A| \geq |B|+1$, then there exist some $a \in A \setminus B$. Let $C = B \cup \{a\}$ be then $|C| = \exp(G)+1$ and we have that $\sigma(C) = \sigma(B) + \sigma(\{a\}) = 0 + a = a = 0 + a = \exp(G)a + a = (\exp(G)+1)a$. Therefore C is a $(\exp(G)+1)$-barycentric subset of A, hence, $BO(\exp(G)+1, G) \leq g(G)$.

Assuming that $A \subseteq G$ such that $|A| = BO(\exp(G)+1, G)$, hence there exists $B \subseteq A$ such that$|B| = \exp(G)+1$, hence $\sigma(B) = (\exp(G)+1)b$ with $b \in B$. Let $C = B \setminus b$ be a $(\exp(G))$-subset of A such that $\sigma(C) = \sigma(B) - \sigma\{b\} = (\exp(G)+1)b - b = \exp(G)b + b - b = 0$. Therefore $C \subseteq A$ is a $(\exp(G))$-subset with a zero-sum, in consequence $g(G) \leq BO(\exp(G)+1, G)$. Therefore, $BO(\exp(G)+1, G) = g(G)$. ■

The following corollary is a consequence of Theorem 6 and Remark 2.

Corollary 6

$BO(3, C_3^2) = 5$, $BO(3, C_3^3) = 10$, $BO(3, C_3^4) = 21$,$BO(3, C_3^5) = 46$ and $112 \leq BO(3, C_3^6) \leq 114$.

The following corollary is a consequence of Theorem 7 and Remark 2.

Corollary 7

$BO(4, C_3^2) = 5$, $BO(4, C_3^3) = 10$, $BO(4, C_3^4) = 21$, $BO(4, C_3^5) = 46$ and $112 \leq BO(4, C_3^6) \leq 114$.

The following corollary is a consequence of Theorem 7 and Remark 2.

Corollary 8 $BO(2n, C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \textit{if } n \textit{ is even,} \\ 2n+3 & \textit{if } n \textit{ is odd.} \end{cases}$

The following corollary is a consequence of Theorem 7 and Theorem 5.

Corollary 9 $BO(2n+1, C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \textit{if } n \textit{ is even,} \\ 2n+3 & \textit{if } n \textit{ is odd.} \end{cases}$

Theorem 8 *([13], Theorem 4.2) Let $p \geq 7$ be an prime number and $\frac{p+1}{2} \leq k \leq p-3$. Then $BO(k, C_p) = k+1$.*

Theorem 9 *([13], Theorem 4.3) Let $p \geq 7$ be an integer prime number and $k = \frac{p-1}{2}$. Then*

$$BO(k, C_p) = \begin{cases} k+1 & \textit{if the multiplicity order of 2 module p is odd} \\ k+2 & \textit{if it is even.} \end{cases}$$

4 Existence conditions of $BO(k, G)$ for elementary 2-groups

Let C_2^m be an elementary 2-group of order 2^m. From the results cited in [13] we have that: $BO(2^m, C_2^m) = 2^m$, $BO(2^m - 1, C_2^m) = 2^m + 1$ and $BO(2^m-2, C_2^m) = 2^m+1$. In this section we study the existence of $BO(k, C_2^m)$ for $3 \leq k \leq 2^m - 3$. In some cases when $BO(k, C_2^m)$ exists, we give its exact value.

The following result is a consequence of Proposition 4 and Corollary 1.

Corollary 10 *Let k be an even integer such that $4 \leq k \leq 2^{m-1} - 1$. Then $BO(k, C_2^m) < 2^m$.*

The following result provides the existence of $BO(2^{m-1}, C_2^m)$.

Theorem 10 $BO(2^{m-1}, C_2^m) \leq 2^m$.

Proof. Assuming that $BO(2^{m-1}, C_2^m) = 2^m + 1$, i.e., each (2^{m-1})-subset A of C_2^m verifies $\sigma(A) \neq 0$ and $\sigma(A^c) \neq 0$. If $0 \in A$, then the $(2^{m-1} - 1)$-subset $B = A \setminus \{0\}$ verifies that $\sigma(B) \neq 0$, that is to say, there is no $(2^{m-1} - 1)$-subset B with zero-sum over C_2^m. Else $0 \in A^c$, then the $(2^{m-1} - 1)$-subset $C = A^c \setminus \{0\}$ verifies that $\sigma(C) \neq 0$; hence, there is no $(2^{m-1} - 1)$-subset C with zero-sum over C_2^m; then a contradiction with Lemma 2. Therefore, $BO(2^{m-1}, C_2^m) \leq 2^m$. ■

The following result gives the existence of $BO(k, C_2^m)$ for even k and $2^{m-1} + 2 \leq k \leq 2^m - 4$.

Theorem 11 *Let k be an even number such that $2^{m-1} + 2 \leq k \leq 2^m - 4$. Then $BO(k, C_2^m) \leq 2^m$.*

Proof. Assuming that $BO(k, C_2^m) = 2^m + 1$, i.e., each k-subset A over C_2^m is not barycentric, in consequence $\sigma(A) \neq 0$ and $\sigma(A^c) \neq 0$. Notice that $|A^c| = 2^m - k$ is an even integer and we have that $4 \leq 2^m - k \leq 2^{m-1} - 1$. So that C_2^m does not contain a $(2^m - k)$-subset A^c with zero-sum, therefore a contradiction with Corollary 10. So that, $BO(k, C_2^m) \leq 2^m$. ■

The following results follow from the last three results .

Corollary 11 *Let k be an even integer such that $4 \leq k \leq 2^m - 4$. Then $BO(k, C_2^m) \leq 2^m$.*

In order to complete the existence of $BO(k, C_2^m)$, we need to show that $BO(k, C_2^m) \leq 2^m$ for all even integers $3 < k \leq 2^m - 3$.

The following result shows the inexistence of $BO(3, C_2^m)$.

Theorem 12 $BO(3, C_2^m) = 2^m + 1$.

Proof. Assuming that $BO(3, C_2^m) \leq 2^m$, that is to say, there exists a 3-subset A in C_2^m such that $\sigma(A) \in A$. Let $B = A \setminus \{\sigma(A)\}$ be a 2-subset in C_2^m such that $\sigma(B) = 0 \in \sum_2 C_2^m$; therefore a contradiction with proposition 4.v. Hence, $BO(3, C_2^m) = 2^m + 1$. ■

The following result gives the exact values of $BO(2^m - 3, C_2^m)$.

Theorem 13 $BO(2^m - 3, C_2^m) = 2^m - 3$.

Proof. Assuming that $BO(2^m - 3, C_2^m) = 2^m + 1$, i.e., each $(2^m - 3)$-subset A over C_2^m is not barycentric, in consequence $\sigma(A) \notin A$; hence $\sigma(A) \in A^c$. Since $\sigma(A) = \sigma(A^c)$, then $\sigma(A^c) \in A^c$. So that, there exists a barycentric

3-subset A^c in C_2^m, that is to say, $BO(3, C_2^m) \leq 2^m$; hence a contradiction with the fact that $BO(3, C_2^m) = 2^m + 1$. Therefore, $BO(2^m - 3, C_2^m) = 2^m - 3$. ■

To finalize the discussion on the existence conditions of $BO(k, C_2^m)$ for the odd integers k in $5 \leq k \leq C_2^m - 5$ we will use the following results:

Proposition 5 *Let k be an even number and $BO(k, C_2^m) = q$. Then $q > k$.*

Proof. Assuming that $BO(k, C_2^m) = k$, i.e., for each k-subset A over C_2^m we have that $\sigma(A) = 0$. Let A be a k-barycentric set over C_2^m such that $0 \in A$ and consider the $(k-1)$-subset $B = A \setminus \{0\}$ over C_2^m, hence $\sigma(B) = \sigma(A) = 0$. Let $0 \neq c \in B^c$ be and consider the k-subset $D = B \cup \{c\}$ over C_2^m, hence $\sigma(D) = \sigma(B) + \sigma(\{c\}) = 0 + c = c \neq 0$. Therefore there exists a non barycentric k-subset D over C_2^m. Hence a contradiction with the fact that $BO(k, C_2^m) = k$. In consequence, $q > k$. ■

Theorem 14 *Let k be an even integer such that $4 \leq k \leq 2^m - 4$. If $BO(k, C_2^m) = q$, then $BO(k+1, C_2^m) = q$.*

Proof. Let A be a q-set over C_m^2 and B a k-subset over A such that $\sigma(B) = 0$. Since $|A| = q > k = |B|$, then there exists $a \in A \setminus B$. Let us consider the set $C = B \cup \{a\}$, notice that it is a $(k+1)$-subset over A such that $\sigma(C) = \sigma(B) + \sigma(\{a\}) = 0 + a = a$ with $a \in C$, that is to say, C is a barycentric $(k+1)$-subset in A. Hence $BO(k+1, C_2^m) \leq q$.

Assuming that A is a subset over C_m^2 such that $|A| = BO(k+1, C_2^m)$, then A contains a $(k+1)$-subset B such that $\sigma(B) = (k+1)b = kb + b = 0 + b = b$ for some $b \in B$. Let $C = B \setminus \{b\}$ be the k-subset of A such that $\sigma(C) = \sigma(B) - \sigma(\{b\}) = b - b = 0$. Hence C is a barycentric k-subset in A, i.e., $q \leq BO(k+1, C_2^m)$. Hence, $BO(k+1, C_2^m) = q$. ■

The following result is a direct consequence of the above theorem.

Corollary 12 *Let k be an odd integer such that $5 \leq k \leq 2^m - 5$. Then $BO(k, C_2^m) \leq 2^m$.*

The following result proves the increase of the values of $BO(k, C_2^m)$ when k is odd and $4 \leq k \leq 2^m - 3$.

Proposition 6 *Let k be an odd integer in $4 \leq k \leq 2^m - 5$. If $BO(k, C_2^m) = q_1$ and $BO(k+1, C_2^m) = q_2$, then $q_1 \leq q_2$.*

Proof. Assuming that $q_2 < q_1$. Let A be a q_2-set over C_2^m and B a barycentric $(k+1)$-subset of A, that is to say, $\sigma(B) = 0$. Let $b \in B$ be and consider the k-subset $C = B \setminus \{b\}$ so that $\sigma(C) = \sigma(B) - \sigma(\{b\}) = 0 - b = -b = b$, i.e., C is a barycentric k-subset of A. Hence $BO(k, C_2^m) \leq q_2 < q_1$, then a contradiction with the fact that $BO(k, C_2^m) = q_1$. Therefore, $q_1 \leq q_2$. ■

The following result is a direct consequence of the above result and Theorem 14.

Corollary 13 *Let k be an integer such that $4 \leq k \leq 2^m - 4$. Then $BO(k+1, C_2^m) \geq BO(k, C_2^m)$.*

5 Conclusions

The goal of the present paper was to continue with the work in [13] for $3 \leq k \leq |G| - 3$. Our present main results are Theorem 1 and Theorem 2. The consequence of these two theorems were the complete existence conditions of cyclic groups and elementary p-groups. Moreover, in Section 4 the existence conditions for elementary 2-groups of our constant $BO(k, G)$ was completely determined. The problem of the existence of $BO(k, G)$ for all abelian groups G remains open, and also the problem of assigning exact values of the k-barycentric Olson constant when $BO(k, G)$ exists; some examples are the interesting results given in Theorem 3 and Theorem 4. The relation between the Harborth and the k-barycentric Olson constants established in this paper could be a good option to provide their exact values.

Acknowledgements

The research of O. Ordaz is supported by the Postgrado de la Facultad de Ciencias de la U.C.V., the CDCH project, and the Banco Central de Venezuela.

References

[1] J. Bierbrauer, Y. Edel, *Bounds on affine caps*, Journal of Combinatorial Designs **10**(2002), no. 2, 111–115. Doi 10.1002/jcd.10000

[2] Y. Edel, S. Ferret, I. Landjev, L. Storme, *The classification of the largest caps in $AG(5,3)$*, Journal of Combinatorial Theory, Ser. A, **99**(2002), no. 1, 95–110. Doi: 10.1006/jcta.2002.3261

[3] W. Gao, A. Geroldinger, *On long minimal zero sequences in finite abelian groups*, Periodica Mathematica Hungarica, **38**(1999), no. 3, 179–211. Doi: 10.1023/A:1004854508165

[4] W. Gao, R. Thangadurai, *A variant of Kemnitz conjecture*, Journal of Combinatorial Theory, Ser. A **107**(2004), 69–86. Doi: 10.1016/j.jcta.2004.03.009

[5] D. Grynkiewicz, *A weighted Erdös-Ginzburg-Ziv theorem*, Combinatorica, **26**(2006), 445-453. Doi: 10.1007/s00493-006-0025-y

[6] Y. Hamidoune, *On weighted sums in abelian groups*, Discrete Mathematics, **162**(1996), 127–132. Doi: 10.1016/0012-365X(95)00281-Z

[7] R. Hill, *On the largest size of cap in $S_{5,3}$*, Atti Accad. Naz. Lincei Rend., **54**(1973), no. 8, 378–384.

[8] R. Hill, *Caps and codes*, Discrete Mathematics, **22**(1978), no. 2, 111–137. Doi: 10.1016/0012-365X(78)90120-6

[9] A. Kemnitz, *On a lattice point probems*, Ars Combinatoria, **16b**(1983), 151–160 .

[10] F. Luca, O. Ordaz, M.T. Varela, *On barycentric constants*, Revista de la Unión Matemática Argentina, **53**(2012), no. 2, 1–12. http://inmabb.criba.edu.ar/revuma/pdf/v53n2/v53n2a01.pdf

[11] L.E. Marchan, O. Ordaz, D. Ramos, W. Schmid, *Some exact values of the Harborth constant and its plus-minus weighted analogue*, Arch. Math., **101**(2013), 501–512. https://arxiv.org/pdf/1308.3315.pdf

[12] A. Mukhopadhyay, *Lower bounds on $m_t(r,s)$*, Journal of Combinatorial Theory, Ser. A **25**(1978), no. 1, 1–13. Doi: 10.1016/0097-3165(78)90026-2

[13] O. Ordaz, A. Plagne, W.A. Schmid, *Some remarks on barycentric-sum problems over cyclic groups*, European Journal of Combinatorics, **34**(2013), no. 8, 1415–1428. Doi: 10.1016/j.ejc.2013.05.025

[14] O. Ordaz, M.T. Varela, F. Villarroel, *k-barycentric Olson constant*, Math. Reports, **11**(2009), no. 61, 33–45. https://www.researchgate.net/publication/265081954_k-barycentric_Olson_constant