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ALGEBRAIC JET SPACES AND ZILBER'S DICHOTOMY IN DCFA

ESPACIOS DE JETS ALGEBRAICOS Y LA DICOTOMÍA DE ZILBER EN DCFA

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Abstract

This is the first of two papers devoted to the proof of Zilber's dichotomy for the case of difference-differential fields of characteristic zero. In this paper we use the techniques exposed in [9] to prove a weaker version of the dichotomy, more precisely, we prove the following: in *DCFA* the canonical base of a finite-dimensional type is internal to the fixed field of the field of constants. This will imply a weak version of Zilber's dichotomy: a finite-dimensional type of *SU*-rank 1 is either 1-based or non-orthogonal to the fixed field of the field of constants.

Resumen

El presente es el primero de dos artículos dedicados a la demostración de la dicotomía de Zilber para el caso de los campos diferenciales de diferencia de característica cero. En éste artículo utilizamos las técnicas desarrolladas en [9] para demostrar una versión débil de la dicotomía: un tipo de dimensión finita y de rango *SU* igual a 1 es modular o no ortogonal al campo fijo del campo de constantes.

1 Introduction and preliminaries

The theory of differentially closed fields (*DCF*) is the model companion of the theory of differential fields. Among the properties of *DCF* we find that it is ω-stable, it eliminates quantifiers and imaginaries. An axiomatization for *DCF* is the following.

Definition 1.1 *Let (K, D) be a differential field. K is differentially closed if and only if K is an algebraically closed field and for every irreducible algebraic variety V, if W is an irreducible algebraic subvariety of $\tau_1(V)$ such that the projection of W onto V is dominant, then there is $a \in V(K)$ such that $(a, Da) \in W$.*

This axiomatization is not the original and most known, but this version, due to Pierce and Pillay ([7]), has a more geometric spirit which will be useful in this paper.

A consequence of ω-stability is Zilber's dichotomy for *DCF*: A type of U-rank 1 is either one-based or nonorthogonal to the field of constants. Proofs and details about *DCF* can be found in [6] and [5].

The theory of difference-differential fields of characteristic zero also has a model-companion, (*ACFA*). It is supersimple, quantifier-free ω-stable and it eliminates imaginaries. It does not eliminate quantifiers and it is not a complete theory, but its completions are easily described. Since *ACFA* is supersimple, its complete types are ranked by the SU-rank. As *DCF*, *ACFA* satisfies Zilber's dichotomy: a type of SU-rank 1 is either 1-based or non-orthogonal to the fixed field. See [3] for proofs of these facts.

The original proofs of these two dichotomies have a heavy use of stability (or simplicity), but recently Pillay and Zigler found more geometric proofs using algebraic jet spaces. One important fact which is key to both proofs is that having finite transcendence degree and having finite rank are equivalent in both *DCF* and *ACFA*.

Hrushovski proved that the theory of difference-differential fields of characteristic zero has a model-companion. We denote it *DCFA*. This theory is supersimple, quantifier-free ω-stable, and it eliminates imaginaries. Proofs of these facts are found in [1]. As *DCFA* is supersimple its types are ranked by the SU-rank, in [2] the author proved that the SU-rank of a model of *DCFA* (that is, the SU-rank of a difference-differential transcendental element) is ω^2, and gives an example (3.1) of a set whose SU-rank 1 but has infinite transcendence degree.

2 Algebraic jet spaces

We list the main properties of jet spaces over algebraically closed fields of characteristic zero. We will suppose all varieties to be absolutely irreducible.

Definition 2.1 *Let K be an algebraically closed field , and let $V \subseteq \mathbb{A}^n$ be a variety over K^n; let a be a non singular point of V. Let $\mathcal{O}_{V,a}$ be the local ring of V at a and let $\mathfrak{M}_{V,a}$ be its maximal ideal.*

Let $m > 0$. The m-th jet space of V at a, $J^m(V)_a$, is the dual space of the K-vector space $\mathfrak{M}_{V,a}/\mathfrak{M}_{V,a}^{m+1}$.

Notation 2.2 *If the variety V is $\mathbb{A}^n$, we write $\mathfrak{M}_a$ instead of $\mathfrak{M}_{V,a}$.*

The following is proved in [9] (Fact 1.2).

Fact 2.3 *Let U, V be irreducible varieties of K^n, $a \in V \cap U$. If $J^m(V)_a = J^m(U)_a$ for all $m > 0$, then $V = U$.*

Proposition 2.4 *Let V be an variety, a a non-singular point of V. Let $\mathcal{O}_a$ be the local ring of V at a, and $\mathfrak{M}_{V,a}$ its maximal ideal. Let*

$\mathcal{M}_{V,a} = \{f \in K[V] : f(a) = 0\}$ be the maximal ideal of the coordinate ring of $K[V]$ of V. Then, for all $m \in \mathbb{N}$, $\mathcal{M}_{V,a}/\mathcal{M}^m_{V,a}$ and $\mathfrak{M}_{V,a}/\mathfrak{M}^m_{V,a}$ are isomorphic K-vector spaces .

PROOF:

This is a consequence from the fact that $\mathcal{M}^i_{a,V} \cap K[V] = \mathfrak{M}^i_{a,V}$ for all i. (cf Proposition 2.2 in [4]) . □

The following fact is proved in [10], Chapter II, section 5.

Fact 2.5 *Let U, V be two irreducible varieties defined over $L \subseteq K$. Let $f : U \to V$ be a finite morphism, and let $b \in V$. If f is unramified at b, then, for any $a \in f^{-1}(b)$ and for any positive integer m, the homomorphism $\bar{f} : \mathcal{O}_{V,b}/\mathfrak{M}^m_{V,b} \to \mathcal{O}_{U,a}/\mathfrak{M}^m_{U,a}$ induced by f is an isomorphism.*

Proposition 2.6 *Let U, V be two irreducible varieties defined over $L \subseteq K$. Let $f : U \to V$ be a dominant generically finite-to-one morphism. Let a be a generic of U over L. Then f induces an isomorphism of K-vector spaces between $J^m(U)_a$ and $J^m(V)_{f(a)}$.*

PROOF:

Since f is separable (as we work in characteristic zero), and since f is dominant and $f^{-1}(f(a))$ is finite, U and V are irreducible and their dimensions are equal, thus f is unramified at $f(a)$. By 2.5, f induces an isomorphism between $\mathcal{O}_{V,f(a)}/\mathfrak{M}^{m+1}_{V,f(a)}$ and $\mathcal{O}_{U,a}/\mathfrak{M}^{m+1}_{U,a}$; whose restriction to $\mathfrak{M}_{V,f(a)}/\mathfrak{M}^{m+1}_{V,f(a)}$ is an isomorphism between $\mathfrak{M}_{V,f(a)}/\mathfrak{M}^{m+1}_{V,f(a)}$ and $\mathfrak{M}_{U,a}/\mathfrak{M}^{m+1}_{U,a}$. Then, by 2.1, f induces an isomorphism between $J^m(U)_a$ and $J^m(V)_{f(a)}$. □

The following lemma (2.3 of [9]) allows us to consider jet spaces as algebraic varieties.

Lemma 2.7 *Let K be an algebraically closed field and V a subvariety of K^n , let $m \in \mathbf{N}$ and let $\mathcal{D}$ be the set of operators*

$$\frac{1}{s_1! \cdots s_n!} \frac{\partial^s}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}}$$

where $0 < s < m + 1$ and $s = s_1 + \cdots + s_n,\ s_i \geq 0$.

Let $a = (a_1, \cdots, a_n) \in V$; and let $d = |\mathcal{D}|$.

Then we can identify $J^m(V)_a$ with

$$\{(c_h)_{h \in \mathcal{D}} \in K^d : \sum_{h \in \mathcal{D}} DP(a) c_h = 0,\ P \in I(V)\}.$$

PROOF:

Let $p : K[X] \longrightarrow K[V]$ such that $Ker(p) = I(V)$; then $p^{-1}(\mathcal{M}_{a,V}) = \mathcal{M}_a$, and $p^{-1}(\mathcal{M}_{a,V}^{m+1}) = \mathcal{M}_a^{m+1} + I(V)$. This gives us the following short exact sequence:

$$0 \longrightarrow (I(V) + \mathcal{M}_a^{m+1})/\mathcal{M}_a^{m+1} \longrightarrow \mathcal{M}_a/\mathcal{M}_a^{m+1} \longrightarrow \mathcal{M}_{a,V}/\mathcal{M}_{a,V}^{m+1} \longrightarrow 0$$

We proceed to describe the dual space of $\mathcal{M}_a/\mathcal{M}_a^{m+1}$: The monomials $(X-a)^s = (X-a_1)^{s_1} \cdots (X-a_n)^{s_n}$ with $1 \leq s_1 + \cdots + s_n = s \leq m$ form a basis for $\mathcal{M}_a/\mathcal{M}_a^{m+1}$, and for each s we have a K-linear map u_s which assigns 1 to $(X-a)^s$ and 0 to the other monomials. The maps u_s form a basis for the dual of $\mathcal{M}_a/\mathcal{M}_a^{m+1}$.

Thus, the dual $J^m(V)_a$ of $\mathcal{M}_{a,V}/\mathcal{M}_{a,V}^{m+1}$, consists of those linear maps $u : \mathcal{M}_a/\mathcal{M}_a^{m+1} \longrightarrow K$ that take the value 0 on $(I(V) + \mathcal{M}_{a,V})/\mathcal{M}_{a,V}^{m+1}$.

Let $f(X) \in K[X]$; applying Taylor's formula we can write, modulo $\mathcal{M}_{a,V}^{m+1}$,

$$f(X) = f(a) + \sum_{1 \leq |s| \leq m} D_s f(a)(X-a)^s,$$

where

$$D_s = \frac{1}{s_1! \cdots s_n!} \frac{\partial^s}{\partial X_1^{s_1} \cdots \partial X_n^{s_n}}$$

If $u = \sum_s c_s u_s$, then u vanishes on $(I(V) + \mathcal{M}_a^{m+1})/\mathcal{M}_a^{m+1}$ if and only if for every $P(X) \in I(V)$, we have

$$\sum_{1 \leq |s| \leq m} D_s P(a) c_s = 0.$$

□

3 Jet spaces in differential and difference fields

We study jet spaces of varieties over differential fields and difference fields. We recall the concepts of D–modules and σ-modules (see [9]).

Definition 3.1 *Let (K, D) be a differential field, and let V be a finite-dimensional K-vector space. We say that (V, D_V) is a D-module over K if D_V is an additive endomorphism of V such that, for any $v \in V$ and $c \in K$, $D_V(cv) = cD_V(v) + (Dc)v$.*

Lemma 3.2 ([9], 3.1) *Let (V, D_V) be a D-module over the differential field (K, D). Let $(V, D_V)^\sharp = \{v \in V : D_V v = 0\}$. Then $(V, D_V)^\sharp$ is a finite-dimensional $\mathcal{C}$-vector space. Moreover, if (K, D) is differentially closed, then there is a $\mathcal{C}$-basis of $(V, D_V)^\sharp$ which is a K-basis of V. (Thus every $\mathcal{C}$-basis of $(V, D_V)^\sharp$ is a K-basis of V)*

Definition 3.3 *A D-variety is an algebraic variety $V \subseteq \mathbb{A}^n$ with an algebraic section $s : V \to \tau_1(V)$ of the projection $\pi : \tau_1(V) \to V$. Then, by 1.1, $(V, s)^\sharp = \{x \in V : Dx = s(x)\}$ is Zariski-dense in V. We shall write $V^\sharp$ when s is understood.*

Proposition 3.4 *A finite-dimensional affine differential algebraic variety is differentially birationally equivalent to a set of the form $(V, s)^\sharp = \{x \in V : Dx = s(x)\}$ where (V, s) is a D-variety.*

Remark 3.5 *Let $V \subseteq \mathbb{A}^n$ be a variety defined over K.*

1. *Given a D-variety (V, s) , we can extend the derivation D to the field of rational functions of V as follows:*
   If $f \in \mathcal{U}(V)$, then we define $Df = \sum \frac{\partial f}{\partial X_i} s_i + f^D$.

2. *If $a \in V^\sharp$ and $f \in \mathfrak{M}_{V,a}$, then $Df(a) = \sum \frac{\partial f}{\partial X_i} s_i(a) + f^D(a) = J_f(Da) + f^D(a) = D(f(a)) = 0$. Thus $\mathfrak{M}_{V,a}$ and $\mathfrak{M}_{V,a}^{m+1}$ are differential ideals of $\mathcal{O}_{V,a}$, so it gives $\mathfrak{M}_{V,a}/\mathfrak{M}_{V,a}^{m+1}$ a structure of D-module over $\mathcal{U}$. Defining $D^* : J^m(V)_a \to J^m(V)_a$ by $D^*(v)(F) = D(v(F)) - v(D(F))$ for $v \in J^m(V)_a$ and $F \in \mathfrak{M}_{V,a}/\mathfrak{M}_{V,a}^{m+1}$, gives $J^m(V)_a$ a structure of D-module.*

Definition 3.6 *Let (K, σ) be a difference field. A σ-module over K is a finite-dimensional K-vector space V together with an additive automorphism $\Sigma : V \to V$, such that, for all $c \in K$ and $v \in V$, $\Sigma(cv) = \sigma(c)\Sigma(v)$.*

Lemma 3.7 ([9], 4.2) *Let (V, Σ) be a σ-module over the difference field (K, σ). Let $(V, \Sigma)^\flat = \{v \in V : \Sigma(v) = v\}$. Then $(V, \Sigma)^\flat$ is a finite-dimensional $Fix\sigma$-vector space. Moreover, if (K, σ) is a model of ACFA, then there is a $Fix\sigma$-basis of $(V, \Sigma)^\flat$ which is a K-basis of V.(Thus every $Fix\sigma$-basis of $(V, \Sigma)^\flat$ is a K-basis of V)*

Remark 3.8 *Let (K, σ) be a model of ACFA. Let V, W be two irreducible algebraic affine varieties over K such that $W \subseteq V \times V^\sigma$, and assume* *that the projections from W to V and V^σ are dominant and generically finite-to-one. Let $(a, \sigma(a))$ be a generic point of W over K. Then, by 2.6, $J^m(W)_{(a,\sigma(a))}$ induces an isomorphism f of K-vector spaces between $J^m(V)_a$ and $J^m(V)_{\sigma(a)}$. We have also that $(J^m(V)_a, f^{-1}\sigma)$ is a σ-module over K.*

4 Jet spaces in difference-differential fields

We describe the jet spaces of finite-dimensional varieties defined over difference-differential fields, and we state the results needed to prove our main theorem 4.8. Finally we give two corollaries: the first is the weak dichotomy, and the second is an application to quantifier-free definable groups.

We start with the definition of a (σ, D)-module.

Definition 4.1 *Let (K, σ, D) be a difference-differential field. A (σ, D)-module over K is a finite-dimensional K-vector space V equipped with an additive automorphism $\Sigma : V \to V$ and an additive endomorphism $D_V : V \to V$, such that (V, D_V) is a D-module over K, (V, Σ) is a σ-module over K and for all $v \in V$ we have $\Sigma(D_V(v)) = D_V(\Sigma(v))$.*

The key point of our proof of 4.8 is the following lemma.

Lemma 4.2 *Let (V, Σ, D_V) be a (σ, D)-module over the difference-differential field (K, σ, D). Let $(V, \Sigma, D_V)^\natural = \{v \in V : D_V(v) = 0 \wedge \Sigma(v) = v\}$ (we shall write $V^\natural$ when D_V and Σ are understood). Then $V^\natural$ is a $(Fix\sigma \cap \mathcal{C})$-vector space. Moreover, if (K, σ, D) is a model of DCFA, there is a $(Fix\sigma \cap \mathcal{C})$-basis of $V^\natural$ which is a K-basis of V. (Thus every $(Fix\sigma) \cap \mathcal{C}$-basis of $(V)^\natural$ is a K-basis of V)*

PROOF:

It is clear that $V^\natural$ is a $(Fix\sigma \cap \mathcal{C})$-vector space. By 3.2 and 3.7 it is enough to prove that there is a $(Fix\sigma \cap \mathcal{C})$-basis of $V^\natural$ which is a $\mathcal{C}$-basis of $V^\sharp$.

Let $\{v_1, \cdots, v_k\}$ be a $\mathcal{C}$-basis of $V^\sharp$, then $\{\Sigma(v_1), \cdots, \Sigma(v_k)\}$ is a $\mathcal{C}$-basis of $V^\sharp$. Let A be the invertible $k \times k$ $\mathcal{C}$-matrix such that $[\Sigma(v_i)]^t = A[v_i]^t$.

Let $\{u_1, \cdots, u_k\}$ be a $\mathcal{C}$-basis of $V^\sharp$. Then there exists an invertible $k \times k$ $\mathcal{C}$-matrix B such that $[u_i]^t = B[v_i]^t$; applying Σ we get $[\Sigma(u_i)]^t = \sigma(B)[\Sigma(v_i)]^t = \sigma(B)A[v_i]^t$. Thus $\{u_1, \cdots, u_k\}$ is in $V^\natural$ if and only if $B = \sigma(B)A$. Since $(\mathcal{C}, \sigma) \models ACFA$, the system $X = \sigma(X)A$, where X is an invertible $k \times k$ matrix, has a solution in $\mathcal{C}$. So we can suppose that $\{u_1, \cdots, u_k\}$ is in $V^\natural$.

Let $v \in V^\natural$, and let $\lambda_1, \cdots \lambda_k \in \mathcal{C}$ such that $v = \lambda_1 u_1 + \cdots + \lambda_k u_k$. Then $v = \sigma(\lambda_1)u_1 + \cdots + \sigma(\lambda_k)u_k$, thus $\lambda_i \in Fix\sigma$ for $i = 1, \cdots, k$. Hence $\{u_1, \cdots, u_k\}$ is a $(Fix\sigma \cap \mathcal{C})$-basis of $V^\natural$. □

Notation 4.3 *Let $(\mathcal{U}, \sigma, D)$ be a saturated model of DCFA. Let $K = acl(K)$ be a difference-differential subfield of $\mathcal{U}$, and let $a \in \mathcal{U}^n$ such that $K(a)_D = K(a)$ and $\sigma(a) \in K(a)^{alg}$.*

Let V be the locus of a over K, and let W be the locus of $(a, \sigma(a))$ over K. Then V^σ is the locus of $\sigma(a)$ over K and the projections $\pi_1 : W \longrightarrow V$ and $\pi_2 : W \longrightarrow V^\sigma$ are generically finite-to-one and dominant.

We set:

$\pi_1^* : K[V] \longrightarrow K[W]$, $F \longmapsto F \circ \pi_1$.

$\pi_2^* : K[V^\sigma] \longrightarrow K[W]$, $G \longmapsto G \circ \pi_2$.

$\overline{\pi_1^*} : \mathfrak{M}_{V,a}/\mathfrak{M}_{V,a}^{m+1} \longrightarrow \mathfrak{M}_{W,(a,\sigma(a))}/\mathfrak{M}_{W,(a,\sigma(a))}^{m+1}$ *the map induced by* π_1^*

$\overline{\pi_2^*} : \mathfrak{M}_{V^\sigma,\sigma(a)}/\mathfrak{M}_{V^\sigma,a}^{m+1} \longrightarrow \mathfrak{M}_{W,(a,\sigma(a))}/\mathfrak{M}_{W,(a,\sigma(a))}^{m+1}$ *the map induced by* π_2^*

$\pi_1' : J^m(W)_{(a,\sigma(a))} \longrightarrow J^m(V)_a$, $w \longmapsto w \circ \overline{\pi_1^*}$.

$\pi_2' : J^m(W)_{(a,\sigma(a))} \longrightarrow J^m(V^\sigma)_{\sigma(a)}$, $w \longmapsto w \circ \overline{\pi_2^*}$.

With respect to the extension of D to the coordinate rings, π_1^* and π_2^* are differential homomorphisms. By 2.6 π_1' and π_2' are isomorphisms of $\mathcal{U}$-vector spaces.

Let $f : J^m(V)_a \longrightarrow J^m(V^\sigma)_{\sigma(a)}$ be the $\mathcal{U}$-isomorphism defined by $f = \pi_2' \circ (\pi_1')^{-1}$.

Since $Da \in K(a)$ there is a rational map $s : V \to \mathcal{U}^n$ such that $s(a) = Da$ and (V, s) is a D-variety. By construction (V^σ, s^σ) and $(W, (s, s^\sigma))$ are also D-varieties.

Lemma 4.4 *$(J^m(V)_a, f^{-1}\sigma, D^*)$ is a (σ, D)-module.*

PROOF:

All we need to prove is that D^* commutes with $f^{-1}\sigma$. Since $f = \pi_2' \circ (\pi_1')^{-1}$ and π_1', π_2' are isomorphisms, and since σ commutes with D^*, it is enough to prove that D^* commutes with π_1' and π_2'.

Let $w \in J^m(W)_{(a,\sigma(a))}$ and $F \in \mathfrak{M}_{V,a}/\mathfrak{M}_{V,a}^{m+1}$.

We want to prove that $D^*(\pi_1'(w))(F) = (\pi_1' \circ D^*(w))(F)$. We have $D^*(\pi_1'(w))(F) = D^*(w \circ \overline{\pi_1^*})(F) = D((w \circ \overline{\pi_1^*})(F)) - w \circ \overline{\pi_1^*}(D(F))$.

On the other hand $\pi_1'(D^*(w))(F) = (D^*(w)\circ\overline{\pi_1^*})(F) = D(w(\overline{\pi_1^*}(F))) - w(D_W(\overline{\pi_1^*}(F))$.

But clearly $D((w \circ \overline{\pi_1^*})(F)) = D(w(\overline{\pi_1^*}(F)))$ and $w \circ \overline{\pi_1^*}(D_V(F)) = w(D_V(\overline{\pi_1^*})(F))$.

The proof is similar for π_2'. □

Lemma 4.5 *Let $K \subseteq K_1 = acl(K_1)$. Let V_1 be the (σ, D)-locus of a over K_1, and let c be the field of definition of V_1. Then $c \subseteq Cb(qftp(a/K_1)) \subseteq acl(K, c)$.*

PROOF:

Clearly $c \subseteq Cb(qftp(a/K_1))$. We know that $a \downarrow_{K,c} K_1$ in DCF , also $\sigma^i(D^j a) \subseteq K(a)^{alg}$; then $acl_{\sigma,D}(K, a) \downarrow_{K,c} K_1$ in ACF, thus $Cb(qftp(a/K_1)) \subseteq acl(K, c)$. □

Remark 4.6 *If we replace a by $(a, \sigma(a), \cdots, \sigma^m(a))$ for m large enough, c and $Cb(qftp(a/K_1))$ will be interdefinable over K(choose m for which the Morley rank of $tp_{DCF}(\sigma^m(a)/K(a, \cdots, \sigma^{m-1}(a)))$ is minimal and for which the Morley degree of $tp_{DCF}(\sigma^m(a)/K(a, \cdots, \sigma^{m-1}(a)))$ is minimal)* .

Lemma 4.7 *Let $K \subseteq K_1 = acl(K_1)$. Let V_1 be the locus of a over K_1. Then $J^m(V_1)_a$ is a (σ, D)-submodule of $J^m(V)_a$.*

PROOF:

Clearly $J(V_1)_a$ is a D-submodule of $J^m(V)_a$. Let W_1 be the locus of $(a, \sigma(a))$ over K_1. Let f_1 be the isomorphism between $J^m(V_1)_a$ and $J^m(V_1^\sigma)_{\sigma(a)}$ induced by the projections from W_1 onto V_1 and $(V_1)^\sigma$; since these projections are the restrictions of the projections from W onto V and V^σ, $f_1 \subseteq f$. So $J^m(V_1)_a$ is a σ-submodule of $J^m(V)_a$. □

Theorem 4.8 *Let $(\mathcal{U}, \sigma, D)$ be a saturated model of DCFA and let $K = acl(K) \subseteq \mathcal{U}$. Let $tp(a/K)$ be finite-dimensional (i.e. $tr.dg(K(a)_{\sigma,D}/K) < \infty$). Let b be such that $b = Cb(qftp(a/acl(K, b)))$. Then $tp(b/acl(K, a))$ is almost-internal to $Fix\sigma \cap \mathcal{C}$.*

PROOF:

By assumption, $trdg(K(a)_{\sigma,D}/K)$ is finite. Enlarging a, we may assume that a contains a transcendence basis of $K(a)_{\sigma,D}$ over K. Then $\sigma(a), Da \in K(a)^{alg}$ and $D^2(a) \in K(a, Da)$. Hence we may assume that $Da \in K(a)$.

Let V be the locus of a over K , W the locus of $(a, \sigma(a))$ over K, thus V^{σ} is the locus of $\sigma(a)$ over K.

Let V_1 be the locus of a over $acl(K, b)$; let b_1 be the field of definition of V_1. By 4.5 $b \in acl(K, b_1)$.

By 4.2 for each $m > 1$ there is a $(Fix\sigma \cap \mathcal{C})$-basis of $J^m(V)_a^{\natural}$ which is a $\mathcal{U}$-basis of $J^m(V)_a$. Choose such a basis d_m such that $d = (d_1, d_2, \cdots) \downarrow_{K,a} b$. Then for each m we have an isomorphism between $J^m(V)_a^{\natural}$ and $(\mathcal{C} \cap Fix\sigma)^{r_m}$ for some r_m. Thus the image of $J^m(V_1)_a^{\natural}$ in $(Fix\sigma \cap \mathcal{C})^{r_m}$ is a $(Fix\sigma \cap \mathcal{C})$-subspace of $(Fix\sigma \cap \mathcal{C})^{r_m}$ and therefore it is defined over some tuple $e_m \subseteq Fix\sigma \cap \mathcal{C}$; let $e = (e_1, e_2, \cdots)$. If τ is an automorphism of $(\mathcal{U}, \sigma, D)$ fixing K, a, d, e, then $J^m(V_1)_a = \tau(J^m(V_1)_a)$; on the other hand, $\tau(J^m(V_1)_a) = J^m(\tau(V_1))_a$, thus for all $m > 1$, $J^m(V_1)_a = J^m(\tau(V_1))_a$ and by 2.3 $\tau(V_1) = V_1$, thus $\tau(b_1) = b_1$ which implies that $b_1 \in dcl(K, a, d, e)$. Hence $b \in acl(K, a, d, e)$. Since $e \subseteq Fix\sigma \cap \mathcal{C}$ and $d \downarrow_{Ka} b$, this proves our assertion. □

As in [8], we deduce the dichotomy theorem.

Corollary 4.9 *If $tp(a/K)$ is of SU-rank 1 and finite-dimensional, then it is either 1-based or non-orthogonal to $Fix\sigma \cap \mathcal{C}$.*

Proof:

We suppress the set of parameters. Let $p = tp(a)$. If p is not 1-based there is a tuple of realizations d of p and a tuple c such that $c = Cb(qftp(d/c)) \nsubseteq acl(d)$. Then $tp(c/d)$ is non-algebraic and by 4.8 it is almost-internal to $Fix\sigma \cap \mathcal{C}$. As $tp(c/d)$ is p-internal we have $p \not\perp Fix\sigma \cap \mathcal{C}$. □

We conclude with an application to definable groups of *DCFA*. We need the following lemmas on quantifier-free stable groups.

Lemma 4.10 *Let M be a simple quantifier-free stable structure which eliminates imaginaries. Let G be a connected group, quantifier-free definable in M defined over $A = acl(A) \subseteq M$. Let $c \in G$ and let H be the left stabilizer of $p(x) = qftp(c/A)$. Let $a \in G$ and b realize a non-forking extension of $p(x)$ to $acl(Aa)$. Then aH is interdefinable over A with $Cb(qftp(a \cdot b/A, a))$. Likewise with right stabilizers and cosets in place of left ones, and $b \cdot a$ instead of $a \cdot b$.*

PROOF:

Let q be the quantifier-free type over M which is the non-forking extension of p. Then aq is the non-forking extension to M of $qftp(a \cdot b/Aa)$. So we must prove that for every automorphism $\tau \in Aut(M/A)$, $\tau(aH) = aH$ if and only if $\tau(aq) = aq$.

Since q is A-definable, $\tau(q) = q$, and $\tau(aq) = \tau(a)\tau(q) = \tau(a)q$. Thus $\tau(aq) = aq$ if and only if $a^{-1}\tau(a)q = q$. But $H = \{x \in G : xq = q\}$, then $a^{-1}\tau(a) \in H$ if and only if $\tau(a)H = aH$, and as H is Aa-definable, $\tau(aH) = \tau(a)H$. □

Lemma 4.11 *Let M be a simple quantifier-free stable structure which eliminates imaginaries. Let G be a connected group, quantifier-free definable in M defined over $A = acl(A) \subseteq M$. Let $c \in G$, let H be the left stabilizer of $qftp(c/A)$ and let $a \in G$ be a generic over $A \cup \{c\}$. Then Hc is interdefinable with $Cb(qftp(a/A, c \cdot a))$ over $A \cup \{a\}$*

PROOF:

We may assume $A = \emptyset$. Let $p = qftp(c/A)$. We know that H is the right stabilizer of p^{-1}, on the other hand, since a is a generic of G we have $c \downarrow c \cdot a$. By 4.10, $Hc \cdot a$ is interdefinable with $Cb(qftp(c^{-1}(c \cdot a)/c \cdot a))$. Since H is $\emptyset$-definable, $Hc \cdot a$ is interdefinable with Hc over a. □

Corollary 4.12 *Let $(\mathcal{U}, \sigma, D)$ be a model of DCFA, and let $K = acl(K) \subseteq \mathcal{U}$. Let G be a finite-dimensional quantifier-free definable group, defined over K. Let $a \in G$ and let $p(x) = qftp(a/K)$. Assume that p has trivial stabilizer. Then p is internal to $Fix\sigma \cap \mathcal{C}$.*

PROOF:

Let $b \in G$ be a generic over $K \cup \{a\}$. By 4.11 a is interdefinable with $Cb(qftp(b/K, a \cdot b))$ over $K \cup \{b\}$ and by 4.8, $tp(Cb(qftp(b/K, a \cdot b))/K, b)$ is internal to $\mathcal{C} \cap Fix\sigma$. Thus $tp(a/K, b)$ is internal to $Fix\sigma \cap \mathcal{C}$; and since $a \downarrow_K b$, $tp(a/K)$ is internal to $Fix\sigma \cap \mathcal{C}$. □

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