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small data existence for the boltzmann equation in L^1 *

existencia de soluciones para la ecuación de boltzmann en L^1 con dato pequeño

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Abstract

An existence theorem for the Boltzmann Equation with force term and small initial data is proved in an L^1 setting.

Keywords: Boltzmann equation, kinetic theory, fixed point.

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Resumen

Se prueba un teorema de existencia de soluciones en el contexto de L^1, para la ecuación de Boltzmann con término fuerza y dato inicial pequeño.

Palabras clave: Ecuación de Boltzmann, teoría cinética, punto fijo.

Mathematics Subject Classification: 35-xx, 82-xx, 82Cxx, 82C40.

1 Introduction

Let us consider the following problem: Find $f(t,x,v) \geq 0$, $f \in L^1(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ Such that

$$\begin{cases} \dfrac{\partial f}{\partial t} + v.\nabla_x f + F.\nabla_v f + t\dfrac{\partial F}{\partial t}.\nabla_v f = Q(f,f) \\ f(0,x,v) = f_0(x,v) = \phi(x,v). \end{cases} \tag{1}$$

The external field

$$\begin{aligned} F :&\mathbb{R}^6 \longrightarrow \mathbb{R}^3 \\ &(t,x,v) \mapsto F(t,x,v) = (F_1(t,x,v), F_2(t,x,v), F_3(t,x,v)) \end{aligned}$$

Differentiable is supposed with respect to the time, it is observed that if F doesn't depend on the time, then (1) decreases to

$$\begin{cases} \dfrac{\partial f}{\partial t} + v.\nabla_x f + F(t,x,v).\nabla_v f = Q(f,f) \\ f(0,x,v) = f_0(x,v) = \phi(x,v) \end{cases} \tag{2}$$

That is the problem considered in the literature, (see [3]), in this sense (1) it is a generalization of (2), here,

$$Q(f,f)(v) = \int_{\mathbb{R}^3} \int_{|w|=1} [w \cdot (v-u)] w [f(u')f(v') - f(u)f(v)] du\ dw,$$

it is the collision operator, being

$$\begin{aligned} v' &= v - [w \cdot (v-u)]w \\ u' &= u + [w \cdot (v-u)]w \end{aligned} \tag{3}$$

u, v, u' and v' are speeds precollision and postcollision respectively and w is a unitary vector, here we use the notation $f(u) = f(t,x,u)$, $f(v) = f(t,x,v)$, etc., and $(w \cdot (v-u)w)$ it is the kernel of Collision Operator.

Let $D = \mathbb{R}^3 \times I_T$, where $I_T = [0, T]$. We define

$$V = \left\{ f(t,x,v) \geq 0 : \frac{\partial f}{\partial t} + v.\nabla_x f + F.\nabla_v f + t\frac{\partial F}{\partial t}.\nabla_v f \in L^1(D \times \mathbb{R}^3), \right.$$
$$\left. f_0(x,v) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

if

$$\left\|f\right\|_V = \left\|f_0\right\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} + \left\|\frac{\partial f}{\partial t} + v.\nabla_x f + F.\nabla_v f + t\frac{\partial F}{\partial t}.\nabla_v f\right\|_{L^1(D\times\mathbb{R}^3)}, \quad (4)$$

then V is a Banach Space, (see[4]).

We will demonstrate that in the space V the problem (1) is well-posed with some additional considerations.

In the theory of the Enskog equation, global existence and continuous dependence on initial data is proved in L^1 for small data. These results are presented in the paper [4]. In the theory of the Boltzmann equation, existence results for the small data in space of functions L^∞ have been studied in [1], [8] and [9], results for the Boltzmann equation in L^1 have been studied for large data in [3]. In the papers [5] and [6] are considered problems of the type (1) close to equilibrium and relativistic near the vacuum respectively.

In this paper we apply the method used for Cercignani in [4] for extend to the Boltzmann equations with force term.

The paper is presented in three sections: In the first one provides the main topics, in the second one, we formulate and prove important results for the demonstration of the main theorems and in third the main theorems are demonstrated.

2 Preliminary

Lemma 2.1 *Let $f \in V$, $f^\sharp(t,x,v) = f(t, x+vt, v+Ft)$ and $g \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Then*

$$|f^\sharp(t,x,v)| \leq g^\sharp(x,v) = g(x+vt, v+Ft) \quad (5)$$

c.t.p. in D. Moreover $||g^\sharp||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = ||f^\sharp||_V = ||f||_V$.

Proof. Deriving $f^\sharp(t,x,v) = f(t, x+vt, v+F)$ with respect to t we obtain

$$\frac{d}{dt}f^\sharp(t,x,v) = f_t + v\cdot\nabla_x f + \left(F + t\frac{\partial F}{\partial t}\right)\cdot\nabla_v f = Q(t, x+vt, v+Ft)$$

then $\quad f(t, x+vt, v+Ft) = f(0,x,v) + \int_0^t Q(s, x+vs, v+Fs)ds, \quad$ i.e.

$$f^\sharp(t,x,v) = \phi(x,v) + \int_0^t Q(s, x+vs, v+Fs)ds,$$

from where

$$\begin{aligned} |f^\sharp(t,x,v)| &= \left|\phi(x,v) + \int_0^t Q(s, x+vs, v+Fs)ds\right| \\ &\le |\phi(x,v)| + \int_0^t |Q(s, x+vs, v+Fs)|ds, \end{aligned}$$

defining $g^\sharp(x,v) = |\phi(x,v)| + \int_0^t |Q(s, x+vs, v+Fs)|ds$, we obtain (5).

Now

$$\begin{aligned} &\|g^\sharp\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |g^\sharp(x,v)|\, dx\ dv \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\phi(x,v)|\, dx\ dv + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |Q(s, x+vs, v+Fs)|\, dx\, dv\, ds \\ &= \|\phi\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} + \|Q^\sharp(t,x,v)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \\ &= \|f^\sharp|_V = \|f\|_V, \end{aligned}$$

moreover $\|f^\sharp|_V \le \|g^\sharp\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$, and so we have that result. ∎

Lemma 2.2 *If* f, $h \in V$ *and*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} ||v-u||^4 du\, dv < \frac{1}{16MT}; \int_{\mathbb{R}^3}\int_{|w|=1} ||v-u|| du\, dv < \frac{1}{8MT} \qquad (6)$$

with M *constant, then* $Q(f,f) - Q(h,h) \in L^1(D\times\mathbb{R}^3)$ *and there exist* $K < 1$ *such that*

$$\|Q(f,f) - Q(h,h)\|_{L^1(D\times\mathbb{R}^3)} \le K\|g^\sharp\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Proof. Show that $Q(f,f)-Q(h,h)$ is bounded,

$$
\begin{aligned}
&\Big\|Q(f,f)-Q(h,h)\Big\|_{L^1(D\times\mathbb{R}^3)} = \\
&= \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\int_{|w|=1}\Big[w\cdot(v-u)\Big]w\Big[\Big(f(v')f(u')-f(u)f(v)\Big)\right.\\
&\qquad \left.-\Big(h(v')h(u')-h(u)h(v)\Big)\Big]du\,dw\right|dv\,dx\,dt\\
&= \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\int_{|w|=1}\Big[w\cdot(v-u)\Big]w\Big[\Big(f(v')f(u')-h(v')h(u')\Big)\right.\\
&\qquad \left.-\Big(h(u)h(v)-f(u)f(v)\Big)\Big]du\,dw\right|dv\,dx\,dt
\end{aligned}
$$

adding and subtracting in the integral the terms $f(v')h(u')$ and $h(u)f(v)$ we obtain that

$$
\begin{aligned}
&\Big\|Q(f,f)-Q(h,h)\Big\|_{L^1(D\times\mathbb{R}^3)} \leq \\
&\leq \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{|w|=1}\|w\cdot(v-u)\|\|w\|\\
&\qquad\Big[|f(v')||f(u')-h(u')|+|h(u')||f(v')-h(v')|\\
&\qquad+|h(u)||h(v)-f(v)|+|f(v)||h(u)-f(u)|\Big]\,du\,dw\,dv\,dx\,dt,
\end{aligned}
$$

apply (5) of lemma 2.1 to the function $f-h\in V$,

$$
\begin{aligned}
&\Big\|Q(f,f)-Q(h,h)\Big\|_{L^1(D\times\mathbb{R}^3)} \leq \\
&\leq R\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{|w|=1}\|v-u\|\Big(g^\sharp(x,u')+g^\sharp(x,v')\\
&\qquad+g^\sharp(x,v)+g^\sharp(x,u)\Big)du\,dw\,dv\,dx\,dt,
\end{aligned}
$$

with $M=\max\{|f(v)|,\,|f(v')|,\,|h(u)|,\,|h(u')|\}$. Using the definition (3) for v' and u' respectively and performing $v'=V,\ u'=U$ such that

$$
\begin{aligned}
V_i &= v_i-\left(\sum_{i=1}^3 w_i(v_i-u_i)\right)w_i\\
&= v_i-w_1w_i(v_1-u_1)-w_2w_i(v_2-u_2)-w_3w_i(v_3-u_3)\quad i=1,2,3
\end{aligned}
$$

and

$$\begin{aligned} U_i &= u_i + \left(\sum_{i=1}^{3} w_i(v_i - u_i) \right) w_i \\ &= u_i + w_1 w_i(v_1 - u_1) + w_2 w_i(v_2 - u_2) + w_3 w_i(v_3 - u_3) \quad i = 1, 2, 3 \end{aligned}$$

we write

$$\begin{aligned} &\left\| Q(f,f) - Q(h,h) \right\|_{L^1(D\times\mathbb{R}^3)} \leq \\ &\leq \; MT \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|v-u\| \, du \, dv \times \\ &\quad \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,U) |J_w(U)| \, dx \, dU + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,V) |J_w(V)| dx \, dV \right) \\ &\quad + MT \int_{\mathbb{R}^3} \int_{|w|=1} \|v-u\| du \, dw \times \\ &\quad \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,v) dx \, dv + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,u) dx \, du \right) \end{aligned}$$

where $|J_w(V)| = |J_w(U)| = 2|w \cdot (v-u)|^3 \leq 2|w|^3 |(v-u)|^3 |\cos^3\theta| = 2|(v-u)|^3$ with $\theta = n\pi,\ n = 0, 1, 2, ...,$ thus

$$\begin{aligned} &\left\| Q(f,f) - Q(h,h) \right\|_{L^1(D\times\mathbb{R}^3)} \leq \\ &\leq \; 2MT \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|v-u\|^4 du \, dv \times \\ &\quad \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,U) dx \, dU + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,V) dx \, dV \right) \\ &\quad + MT \int_{\mathbb{R}^3} \int_{|w|=1} \|v-u\| du \, dw \times \\ &\quad \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,v) dx \, dv + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g^\sharp(x,u) dx \, du \right) \end{aligned}$$

and for (6) we have that

$$\left\| Q(f,f) - Q(h,h) \right\|_{L^1(D\times\mathbb{R}^3)} \leq K \, \|g^\sharp\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}. \quad \blacksquare$$

3 Main theorems

Theorem 3.1 *Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and*

$$\|f_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq C_0$$

with $C_0 > 0$ constant. Then there exist a unique solution of (1).

Proof. Given $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $h \in V$, if f is the solution of

$$\begin{cases} f_t + v.\nabla_x f + F.\nabla_v f + t\dfrac{\partial F}{\partial t}.\nabla_v f = Q(h,h) \\ f(0,x,v) = f_0(x,v) \end{cases} \tag{7}$$

we define the operator $N : h \in V \rightarrow N(h) = f \in V$ so according to the norm for the space V defined in (4) $\|f\|_V = \|N(h)\|_V$, as f is solution of the equation (7),

$$\begin{aligned} \Big\|N(h)\Big\|_V = \Big\|f\Big\|_v &= \Big\|f_0\Big\|_{L^1} + \Big\|f_t + v.\nabla_x f + F.\nabla_v f + t\frac{\partial F}{\partial t}.\nabla_v f\Big\|_{L^1} \\ &= \Big\|f_0\Big\|_{L^1} + \Big\|Q(h,h)\Big\|_{L^1}. \end{aligned}$$

Thus for lemma 2.1, since $h \in V$,

$$\|N(h)\|_V \leq C_0 + \|g^\sharp\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = C_0 + \|h\|_V.$$

Defining $B_R = \{h \in V : \|h\| \leq R\} \subset V$, then N mapping the ball B_R in another ball $B_{R'}$ being $R' < c_0 + R$. Considering h and h' in B_R, $f = N(h)$ and $f' = N(h')$ in $B_{R'}$ we prove that N is Lipschitz continuous. f and f' are such that

$$\begin{aligned} f_t + v\cdot\nabla_x f + F\cdot\nabla_v f + t\frac{\partial F}{\partial t}\cdot\nabla_v f = Q(h,h) \\ f'_t + v\cdot\nabla_x f' + F\cdot\nabla_v f' + t\frac{\partial F}{\partial t}\cdot\nabla_v f' = Q(h',h'), \end{aligned}$$

thus

$$(f-f')_t + v.\nabla_x(f-f') + F.\nabla_v(f-f') + t\frac{\partial F}{\partial t}.\nabla_v(f-f') = Q(h,h) - Q(h',h'),$$

with $(f-f')_0(x,0) = 0$. Using the lemmas 2.1 and 2.2 we obtain that

$$\|f - f'\|_v \leq K\|g^\sharp(x,v)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = K\|h - h'\|_V.$$

Thus there exist a unique $h \in V$ that is solution of (1). ■

Theorem 3.2 *If the initial data* $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* $f_0^{'} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *are such that* $f_0 \neq f_0^{\sharp}$ *and satisfy* $||f_0||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq C_0$, $||f_0^{'}||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq C_0$. *then*

$$\|f - f^{'}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq L\|f_0 - f_0^{'}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$$

with L *a constant different from zero.*

Proof. As the initial data are different, using the lemmas 2.1 and 2.2 we obtain

$$\|f - f^{'}\|_V \leq \|f_0 - f_0^{'}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} + K\|h - h^{'}\|_V$$

if $f = h$, then

$$\|f - f^{'}\|_V - K\|f - f^{'}\|_V \leq \|f_0 - f_0^{'}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$$

where

$$\|f - f^{'}\|_V \leq \frac{1}{1-K}\|f_0 - f_0^{'}\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}. \quad \blacksquare$$

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