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A DICHOTOMOUS PROPERTY OF THE TOTAL VARIATION OF A PROCESS WITH INDEPENDENT INCREMENTS

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Abstract

We establish a property for the total variation of a cad-lag process with independent increments which is dichotomous in the sense that only two alternatives are possible. For this purpose we introduce the methods of nonstandard analysis with the study of PII processes in near intervals. Finally we discuss,in the case of continous processes, an equivalent condition for one of the alternatives of the main theorem .

Keywords: total variation of a PII; PII in near intervals; additive decompositions; L^2-regular martingales; continuous shadow theorem; nearby processes.

Resumen

Establecemos una propiedad para la variación total de uin proceso de incrementos independientes, que es dicotómica en el sentido de que solo dos alternativas son posibles. Para este efecto introducimos los métodos del análisis no estándar con el estudio de los procesos PII en casi-intervalos. Finalmente discutimos, en el caso de procesos continuos, una condición equivalente para una de las alternativas del teorema principal.

Palabras clave: variación total de un PII, PII en casi intervalos, descomposiciones aditivas, martingalas L^2-regulares, teorema de la sombra continua, procesos "nearby".

Mathematics Subject Classification: 60G20, 60G17

1 Introduction

Let Y be a process with independent increments (PII process), not necessarily homogeneous, defined on a compact interval I of $\mathbb{R}$, continuous in probability and whose trajectories are rigth-continous and limited to the left (the so called cad-lag property) . We consider

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the variable $V(Y)$ defined by the variation of Y in I : $V(Y) = \sup_\rho \sum |Y(t_{i+1}) - Y(t_i)|$ where the supremum value is taken over all partitions $\rho = \{t_i\}$ of I. This variable takes values in $\mathbb{R}_+ \cup \{\infty\}$. The aim of this paper is to establish the following result:

Theorem 1 *There are only two possible alternatives*

1. *$V(Y)$ is a.s. finite.*
2. *$V(Y)$ is a.s. infinite.*

The problem of the behavior of the variable $V(Y)$ has been analysed by many authors in the framework of processes with independent increments. For example a classical result about brownian motion states that its variation is a.s. infinite over all compact interval, while counting PII processes, in particular Poisson processes, are examples on which alternative 2) is true. The special case of homogeneous PII has been studied by in [10], [11] by Millar, Bretagnolle and in [12] , [13] the same authors were able to prove an analogous result for the so called p-variation of a Lévy process $(1 < p < 2)$. The non-homogeneous case is less common, although in [9] there are given sufficient conditions for alternative 1) in terms of the characteristics of the processes. It seems however that in all these works the simple alternatives given in the theorem above has not been established in the non homogeneous case we treat it here.

Studying the total variation of a stochastic process is in general a hard task, even in the case of regularity of cad-lag processes. It seems that difficulties arise because of the possibility for $V(Y)$ to take on infinite values in non-negligeable sets, and the consequent problems to apply classical results of integration theory. A way to bypass the classical reasoning in this problem is to resort to nonstandard analysis methods. In fact, many ideas coming from nonstandard analysis have been used for a long time to treat continuous time problems in analysis. In the stochastic case, the nonstandard analysis viewpoint has allowed the possibility of treating processes in continuous time by means of its discrete time version, in the setting of finite probability spaces and finite indexed processes in near intervals (see [8] for example). In this paper I propose a study based on the preceding ideas for certain processes in near intervals (named also PII), which are the discrete analogue of processes with independent increments.

In the first section we give the necesary results on PII in near intervals, that will be used in the next section, where the proof of the theorem is made by nonstandard arguments such as the transfer principle. The study of section 1 is completely autonomous from the rest and can be viewed as a piece of a general theory for PII processes in near intervals which I am trying to develop in many ways ([5],[6],[7]). The framework adopted for the nonstandard analysis is that of Internal Set theory (ISt),whose exposition can be found in [1] or [4], and Nelson's theory in [2] for finite probability spaces (partly repeated in [3]).

2 Preliminary results on PII processes in near intervals

We recall some notations and general definitions on the nonstandard analysis theory of finite probability spaces. The following terminology and notation of will be used: a real number x is infinitesimal if $|x| < a$ for all standard $a > 0$. It is called unlimited if $|x| > a$ for all real standard a. If x, y are real numbers, we denote : $x \approx y$ in case $x - y$ es infinitesimal, $x \approx \infty$ in case x is unlimited, $x \ll \infty$ in case x is not an unlimited positive real number.

In the finite version we are dealing with, a finite space of probability (Ω, P) is given, a **near interval** T of the line (that is a finite set of points of $\mathbb{R} : t_0 = a < t_1 < \ldots < t_n = b$ with the property that contiguous points are infinitelly close). We call length of t the value $(b - a)$. By $t + dt$ the right contiguous point of t is denoted. For a given process $X = (X_t, t \in t)$ in (Ω, P) indexed by T the *increment* en t is $dX_t = X_{t+dt} - X_t$. A *filtration* en (Ω, P) means a nondecreasing sequence $F = (F_t, t \in T)$ of algebras of random variables. The *natural filtration* of X is the one in which F_t is the algebra generated by the de variables $(X_u, u \leq t)$. We say that a process X is of *F-independent- increments* (F-PII) if for each t the variable X_t belongs to F_t and the increment dX_t is independent of each variable of F_t. Therefore if the variables $(dX_t, t \in T, t < b)$ are independent X is an F-PII for natural filtration of X.

The proceses X is of *uniformly limited increments* (IUL) if there exists a real limited positive number c such that $|dX_t| \leq c$ everywhere. A point t in T is a fixed discontinuity of X if X is not a.s. continuous in T. Let us abreviate fixed discontinuity point by f.d.

The process whose increments are of the form $|dX|$ and that is null in a is called the *total variation process* of X, and its value in each t of T is the *total variation of X in* $[a, t] \cap T$.

Two complex valued functions f, g defined in t are said to be equivalent if $f(t) \approx g(t)$ for all t in T, and this case is denoted by $f \approx g$ (it is understood that the relation is relative to the near interval T).

2.1 L^2 martingales

Martingales of F are said to have the L^2 -property (they are called L^2 -martingales), if the variable X_b is L^2-integrable in the sense of Loeb-Nelson. For such martingales it is true that every variable X_t is also L^2-integrable. Indeed, X^2 being a submartingale then $X_t^2 \leq E_t\left(X_b^2\right)$, and $E_t\left(X_b^2\right)$ is also L^2-integrable (Theorem 8.3 en [2]). An F-PII martingale X is L^2-regular in t if X is L^2 and the variance function of X is continuous in T. By theorem 12.3 in [2] concerning regularity properties of martingales, every L^2-regular martingale is a.s. of limited fluctuation in T, and by 11.4 in [2] it does not posses fixed discontinuities.

Martingales of F of the F-PII type are particularly important . The next theorem gives sufficient conditions for the L^2 and L^2-regularity properties for martingales of this type:

Theorem 2 *Let X be a F-PII martingale satisfying IUL condition and null in a. Then:*

a) X is L^2- integrable $\Leftrightarrow$ its variance function is limited in T

b) if T is of limited length: X is L^2-regular $\Leftrightarrow$ X does not posses f.d.

Proof: a) The implication $\Rightarrow$ is inmediate. For the other it is sufficient that for some $p \ll 2$ we have $E(|X_b|^p) \ll \infty$ (see [2] on the general theory Loeb-Nelson integral). We will show that this happens for $p = 4$. By hypotesis there exists a limited real c such that $|dX| \leq c$.
Every martingale null in a is of orthogonal increments, and X being moreover the sum of the centered independent increments dX_t when expanding X_b^4 we obtain:

$$E(X_b^4) = 3\sum_{s\neq t} E(dX_s^2 dY_t^2) + \sum_s E(dX_s^4)$$

The second term is bounded above by $c^2 \sum_s E(dX_{s^2}) = c^2 Var(X)_b \ll \infty$. The first term is also limited by PII hypotesis:

$$\begin{aligned}
\sum_{s\neq t} E(dX_s^2)E(dX_t^2) &= \sum_s E(dX_s^2) \sum_{t:t\neq s} E(dX_t^2) = \sum_s E(dX_s^2)(Var(X)_b - E(dX_s^2)) \\
&= Var(X)_b \sum_s E(dX_s^2) - \sum_s E(dX_s^2)^2 \\
&= Var(X)_b^2 - \sum_s (E(dX_s^2))^2
\end{aligned}$$

and it is sufficient to remark that $\sum_s (E(dX_s^2))^2$ is bounded above by $c^2 Var(X)_b$.

b) The implication "$\Rightarrow$" follows from the comments above. For $\Leftarrow$ suppose now that X does not have f.d. and $Var(X)_b \ll \infty$. From a) X is L^2. If s, t are infinitely close pointes in T, in the absence of f.d., we have $X_t - X_s \approx 0$ a.s. and because $X_t - X_s$ is L^2, it follows by the Lebesgue theorem that $Var(X)_t - Var(X)_s = \|X_t - X_s\|_2 \approx 0$, that is the continuity of $Var(X)$ en T. Hence X es L^2-regular en T. It is then sufficient to prove that the varaition function is limited in T under the conditions on de f.d.

Observe that the increments of the variance function of X are infinitesimal. Indeed for every t the variable $(dX_t)^2$ is a.s. infinitely close to 0 and is L^1-integrable by the IUL condition and then the Lebesgue theorem (non standard version) we have $E((dX_t)^2) \approx 0$. Let us suppose by contradiction that $Var(X)_b \approx 0$. Then there exist points s, t in T, $s < t, s = t$ such that $0 \ll Var(X)_t - Var(X)_s \ll \infty$ (see remark after the proof.) The process $X - X_s$ is a PII martingale on the near interval $T \cap [s, t]$ with variance function $Var(X) - Var(X_s)$. Because $Var(X)_t - Var(X)_s \ll \infty$. by a) shows that $X - X_s$ is L^2 in this near interval. But besides $X_t - X_s$ is a.s. infinitely close to 0 (hypotesis on f.d.) and by the Lebesgue theorem $\|X_t - X_s\|_2 \approx 0$, a relation that contradicts the fact that $Var(X)_t - Var(X)_s$ is not infinitesimal. ∎

Remark: The above mentioned result can be justifiied by a general property of functions defined in near intervals T, given by the following two statements:

1) If the length of T is limited then a subset of T of unlimited cardinality whose points are all non infinitely close does exist. This fact can be easily deduce by a contradiction argument.

2) Let f be a function in T. Given two points s,t in T, we call the quantity $|f_s - f_t|$ variation of f in s,t. Let us suppose that increasing in T such that $f_b \approx \infty$ and all of its increments are infenitesimal. Then for every positive and limited real ε there exists a variation of f in two infinitely close points of T that is limited and $\geq \varepsilon$. Indeed for this later statement it suffices to prove the assertion for limited and non infinitesimal ε. Let n be the greatest integer such that $f_b \geq \varepsilon$. Then n is unlimited. If $p \leq n$, define t_p as the least t in T such that $f_{t+dt} \geq p\varepsilon$. Then since $f_{tp} < p\varepsilon$ and $df_{tp} \approx 0$ it follows: $f_{tp} \approx p\varepsilon$. Hence the points t_p, p$\leq n$, define a stricly increasing sequence in T. But T being of limited length, by the statement 1) we deduce that there exist $i \neq j$, $i < j$ with $t_i \approx t_j$, and a fortiori $t_i \approx t_{j+1}$. Consequently $f_{ti+1} - f_{ti} \approx \varepsilon$. and then $f_{ti+1} - f_{ti}$ is limited, and since $\varepsilon, \varepsilon/2$ are not inf. close we have $f_{ti+1} - f_{ti} > \varepsilon/2$.

Finally the assertion made in the proof of the theorem follows easily by applying assertion 2) the function $f(u) = Var(X_u) - Var(X_s)$ in the near interval $T \cap [s,t]$.

2.2 Additive decomposition of PII processes in near intervals

Let(Ω, P) be a finite probability space endowed with a filtration F, a near interval T, and a stochastic process X over T. There exists a canonical way to decompose X as the sum of its previsible part and its martingale part (see [2], chap. 9), denoted respectevily by X^p, $X^{\hat{}}$ and defined as follows: $dX_t^p = E_t(dX_t), X_0^p = 0, X^{\hat{}} = X - X^p$. We have $X = X^p + X^{\hat{}}$and wecall it the additive decomposition of X. If X is an F- martingale, $X^2$is a submartingale of F. The previsible part of X^2 is denoted $[X]$ and then $d[X]_t = E_t(d(X_t)^2) = E_t((dX_t)^2 + 2X_t dX_t) = E_t((dX_t)^2)$. Let the expectation function of X be the function over T denoted by $E(X)$ and such that $E(X)_t = E(X_t)$ and the variance function of X, denoted $Var\,(X)$, be the function over t defined by $Var\,(X)_t = Var(X_t)$. Then for every t we have

$$dVar\,(X)_t = Var(dX_t), d(EX)_t = E(dX_t).$$

In the particular case of processes with the PII property it is readily seen that the previsible part is a deterministic function over t and equals its expectation function, and in the other hand the previsible part of the square of its martingale part is the variance function of the same function. Hence the martingale part of a PII process is also a PII process.

We will analyse later the additive decomposition for PII processes having regularity properties. For this purpose it is convenient to introduce the processes of the form $S(f(dX))$, for one variable functions f, and defined as: $S(f(dX))_a = 0, dS(f(dX))_t = f(dX_t)$. In the sequel we consider the truncation functions $f(X) = X^(\varepsilon) = X1_{\{|X| \leq \varepsilon\}}$ whose associated $S(f(dX))$ process is $S(dX^{(\varepsilon)})$. The processes $X - X_a - S(dX^{(\varepsilon)}) = S(dX - dX^{(\varepsilon)})$ (resp. $S(dX^{(\varepsilon)})$) give for every t the sum of the increments of X larger in absolute value than ε (resp. smaller than) before time t . For the absolute value function $f(X) = |X|$ we obtain $S(|dX|)$, that is the total variation of X, already mentioned in the introduction of this section.

Properties of the $S(f(dX))$ processes:
Let X be an F-PII. For every function f the process $S(f(dX))$ is an F-PII over T. If moreover X is of limited fluctuation a.s. over T, then for every $\varepsilon \gg 0$ the processes $S(dX - dX^{(\varepsilon)})$ is a.s. of limited fluctuation , and the fixed discontinuities of the processes $S(dX - dX^{(\varepsilon)})$ are themselves fixed discontinuities of X.
Proof: The increment $S(f(dX_t))$in t being a function of dX_t it follows that this variable is independent of F_t, hence the first result. Let X be of limited fluctuation a.s. and $\varepsilon \gg 0$. Because a non null increment of $S(dX - dX^{(\varepsilon)})$ in t means that $|dX_t| > \varepsilon$, this process must have a.s a limited number of non null increments in T, for otherwise there would exist an unlimited number of ε-fluctuations of X over T. Hence the second assertion.
For the statement concerning fixed discontinuities we first remark that if t is a f.d. . of $S(dX - dX^{(\varepsilon)})$ then with non infinitesimal probability there exists an $s \approx t$ such that the values $S(dX - dX^{(\varepsilon)})_t$ and $S(dX - dX^{(\varepsilon}\ )_s$ are not equivalent . In particular, there must exist an $s' \approx t$ such that $(dX - dX^{(\varepsilon)})_{s'} \neq 0$, $|dXs| > \varepsilon$, and so t is a f.d. of X. ∎

The additive decomposition of certain processes has an important special property:

Theorem 3 *Let X be an F-PII a.s. of limited values in t of limited length, without f.d. and satisfying IUL condition. then its expectation function is continuous and limited in t and its associated martingale is L^2-regular in T.*

Proof: We use an enlargement of the original space, more precisely let $(\Omega \times \Omega, P \times P)$ be the product space and therein a process X independent of X with the same law (we can put for example $X'(w, w') = X(w')$ and define $Z = X - X'$. It is inmediate that Z satisfy the same conditions as X and besides $E(dZ_t) = 0$, hence Z is also a martingale. the process Z satisfy conditions b) of theorem 2 and from this theorem it follows that $Var(Z)_b \ll \infty$, so $Var(Z)_t \ll \infty$ for every t. But $Var(Z_t) = Var(X_t) + Var(X'_t)$, and then $Var(X_t) \ll \infty$. In particular $X_t - E(X_t)$ is a.s. limited and the same being true for X_t we deduce that $|E(X_t)| \ll \infty$. Hence $E\left|X_t^2\right| \ll \infty$. In particular the variables X_t are all $L^1$and in the absence of f.d. the expectation function of X is continuous. Hence the martingale associated to X does not have f.d. as the sum of two processes without f.d., then it satisfies conditions b) of the same theorem, from which we infer that it is L^2-regular in t. ∎

2.3 A dichotomous property for the total variation

We consider the total variation process of a PII process X in a near interval t of limited length, a.s. of limited values and of limited fluctuation, without f.d. (see introduction of section 1 for the definition of the total variation) :

Theorem 4 *The total variation of X in t must satisfy only one of the following exclusive cases :*

1) it is a.s limited,

2) it is a.s. unlimited.

The case 1) being true if $S(E(|dX|^{(\delta)}))_b \ll \infty$ for any real d such that $0 \ll d \ll \infty$. In case 1) and if additionally X has infinitesimal increments everywhere, then X is equivalent to its expectation function.

PROOF: Let δ be a positive real number, $0 \ll \delta \ll \infty$. Then the process $S(dX - dX^{(\delta)})$ is a.s. the sum of a limited number of limited terms and so it is a.s. of limited total variation in T. Because the total variation of X is the sum of the total variations of $(S(dX - dX^{(\delta)}))$ and $S(dX^{(\delta)})$ it is sufficient to consider the process $S(dX^{(\delta)})$. This one is a PII on T, without f.d. (see property above) and clearly has the IUL property. then we can assume that the process X has moreover the IUL property. Let Z be total variation process of X and $Z\hat{}, X\hat{}$ be the martingales associated to Z, X respectively. We know by their definition that $d(Var(Z\hat{})_t = Var(d(Z)_t)$, $d(Var(X\hat{})_t = Var(dX_t)$, and since $dZ_t = |dX_t|$, and from the fact that $Var(Y) \geq Var(|Y|)$ holds for every random variable Y, we obtain $d(Var(Z\hat{})_t \leq d(Var(X\hat{})_t$. Since the function $Var(X\hat{})$ is continuous and limited in T (by theorem 3), by the last inequality it follows that the same is true for $Var(Z\hat{})$. But $Z\hat{}$ satisfies IUL condition and by b) of theorem 2 it follows that $Z\hat{}$ is L^2-regular. A fortiori $Z\hat{}_b = Z_b - E(Z_b)$ is a.s. limited, and hence the first two statements of the theorem are obtained. In the case 1) we have $S(E(|dX|^{(\delta)}))_b \ll \infty$ for every real d such that $0 \ll \delta \ll \infty$, and a fortiori this property holds for any infinitesimal positive δ. Let now be X with infinitesimal increments everywhere. there exists an infinitesimal ε such that $|dX| \leq \varepsilon$ everywhere, and hence$X = S(dX^{(\varepsilon)})\hat{} + E(S(dX^{(\varepsilon)}))$. If we put $C = S(dX^{(\varepsilon)})\hat{}$ we have (for every $0 \ll \delta \ll \infty$): $Var(C) \leq S(E((dX^{(\varepsilon)})^2) \leq \varepsilon S(E(|dX|^{(\varepsilon)})) \leq \varepsilon S(E(|dX|^{(\delta)}))$, and the right hand side expression being the product of an infinitesimal and a limited number we infer that $Var(C)_b \approx 0$. By the classical maximal inequality for L^2 - martingales we have that a.s. $\max_{t \in T} |Ct| \approx 0$, which means that C is a.s. equivalent to the null process, and then the last statement. ∎

3 Proof of the main theorem and search for equivalent conditions

After the results of section 1 we are able to give a nonstandard proof of the main theorem. Let then Y be as in the main theorem. In order to translate results of the last section into classical terms we have to consider a nearby process Y' of Y, Y being a standard process, defined in a near interval T of I (we refer to the appendix to recall of the notion of nearby process and the equivalence theorems that will be used in the sequel). It is a consequence of E1 and E2 in the appendix that Y' is a PII a.s of limited fluctuation and without f.d. Moreover, we can assert that Y' is a.s. of limited values: a.s. $\max_{s \in T} |Y'_s|$ is limited. These properties of Y' allow one to apply theorems of section 1 to this process. We are now able to establish the main theorem:

PROOF OF MAIN THEOREM: The result expressed in this theorem being a classical property, it can be assumed by transfer that Y is an standard process defined in an standard probability space. We consider then its nearby process Y' defined in a near interval T of I which is a PII, a.s of limited fluctuation, without f.d. and a.s. of limited values.

The equivalence results E4 and E4' and the dichotomous property for processes in near intervals of the theorem of section 1-3 applied to the process Y', gives then the answer. ■

It would be desirable to find equivalent conditions that ensure one of the alternatives given in the main theorem, in terms of the triplet $(B, V, \prod)$ that determines the Lévy-Khintchin formula for the increments of the process Y. We recall that in this triplet B and V are continous functions in I, V being nondecreasing and positive, and denotes the Lévy measure of the jump measure of Y. In [9] it is shown that in order that the total variation of Y in I be a.s. limited (case 1) of main theorem), it is necessary and sufficient to have the following conditions:

$$B \text{ is of finite variation in } I, V = 0, \int_{I\times[0,1]} \ln|X|\pi(dt, dX) < \infty.$$

We are going to prove the preceding result only for continous processes Y and, as it was done for the main theorem, by means of the nonstandard analysis methods. Therefore, it is enough to consider the pair (B, V), the measure $\prod$ being a null measure. In this case the condition above is equivalent to saying that Y is equivalent to its expectation function B, which is of bounded variation in I. The proof is based in the condition given in the theorem 4.

We use a corollary of the so called *continuous shadow theorem* that states that in case I is compact and standard then for every function $f' : T \to \mathbb{R}$, limited and S-continuous, there exists a function $f : I \to \mathbb{R}$, standard and continuous, such that $f'(t) \approx f(t)$ for every $t \in T$. The reader is referred to [4] for example, for details about this important result. We denote by $\Phi_Z(u)$ the value in u of the characteristic function of some variable Z, and by S we refer to the S-continuity property of the exponential function for limited values of the argument.

Theorem 5 *Let Y be a continous PPI in a compact interval I. Then Y is a.s. of bounded variation in I if and only if Y is equivalent to its expectation function B and this one is of bounded variation in I.*

Proof: It is sufficient to prove the necessary condition , the other implication being trivial. We suppose then that Y is a.s. of bounded variation in I. By transfer we can assume that Y is standard. In this case its nearby process Y' is a.s. continous in T and has infinitesimal increments everywhere (E3' in appendix). By E4 the condition on Y implies that Y' is a.s. of limited variation in T. But by the last part of the theorem 4 it is true that Y' is equivalent to its expectation function B. Let u be a fixed limited number and let $s < t$ fixed in T. Thanks to S it is true that: a.s. $exp(iu(Y'_t - Y_s)) \approx \exp(iu(B'_t - B_s))$. In this relation the Lebesgue theorem is applicable because both variables are limited and so $\Phi'_{(Y'_t - Y_s)}(u)) \approx \exp(iu(B'_t - B_s))$. The function B' being S-continous in T, then there exists a standard continous function B in I equivalent to B' in T. By S we have $\exp(iu(B'_t - B'_s)) \approx \exp(iu(B_t - B_s))$. On the other hand, by definition of nearby process, we have $\Phi_{(Y'_t - Y'_s)}(u) \approx \Phi_{(Y_t - Y_s)}(u)$ for all limited u, and then $\Phi_{(Y_t - Y_s)}(u) \approx \exp(iu(B_t -$

$B_s)$). In this last relation both members are standard if s, t, u are standard, hence they are equal in this case. But T contains all the standard reals of I and therefore the relation $\Phi_{(Y_t - Y_s)}(u) = \exp(iu(B_t - B_s))$ is valid for all s, t standard in I, and u standard. By transfer we deduce that preceding equality holds for all s, t in I and u in $\mathbb{R}$, and since the right hand member defines the characteristic function of the constant $B_t - B_s$, it follows that Y is equivalent to this function. ∎

We wonder whether the reasoning made in the proof of the theorem above can be applied to the more general case of discontinous PII, and by this way deducing the general equivalent conditions stated in [9]. this would necessarily require a rigorous nonstandard interpretation of the characteristic triplet of a PII , actually in terms of its nearby process. But we still lack of such an interpretation, the main difficulty being that of the Lévy measure of the jump random measure. However the author has obtained some improvements in this problem by finding approximated formulas of the Lévy-Khintchin type for the characteristic function of a PII in near intervals (see [6]).

Appendix: Recalls on nearby processes and equivalence theorems

We first recall some general results concerning the relations between PII process in near intervals and the classical notion in the continous time setting. In the framework of the IST, the discrete versions of classical processes are named nearby processes after the terminology adopted in [2]. In the case of continuous time they are formalized as follows: given a standard process Y_t, $t \in I$, I interval of $\mathbb{R}$, defined in a standard space (Ω, A, P), a nearby process to Y is one Y'_t, $t \in T$, t a near interval containing all the standard points of I, defined in a finite space (Ω', P), subspace of (Ω, A, P), that satisfies the relation $\sum_{t \in T} |Y_t - Y'_t| \approx 0$, except in an event of infinitesimal probability. We may make the definition precise adding that for every positive $\varepsilon \approx 0$ the process Y' may be chosen such that the relation:

$$\sum_{t \in T} |Y_t - Y'_t| \leq \varepsilon, \qquad (*)$$

holds except in the event of infinitesimal probability. It is shown that such a nearby process Y always exists ([2], theoren A.1 in appendix).

The construction of Y' given in [2] can be slightly modified so that PII property is conserved for Y in case the standard process Y is PII. We give in detail such modification: let T be a near interval containing all the standard points of I. In the same way as in [2] we construct a nearby process to the process of the increments of Y in T: $((dY)'_t, t \in T)$ is nearby to $(dY_t, t \in T)$. Relation (*) holds for some ε. In the proof of this fact it is worth noting that the nearby process is defined by a relation of the form $(dY)'_t = f(dY_t)$ for every t, where f is a real valued function. Then thanks to condition PII the variables $((dY)'_t, t \in T)$ are independent too. Let us choose ε in such a way that $\varepsilon|T|$ is infinitesimal and define $Y'_t = \sum_{s \in T, s<t} (dY)'_s$. Since except for a rare event we have $|Y_t - Y'_t| \leq \sum_{s \in T, s<t} |(dY)'_s - dY_s| \leq \varepsilon$ then $\sum_{s \in T, s<t} |Y'_s - Y_s| \leq \varepsilon|T| \approx 0$ occurs except in this event. The process Y' is then nearby to Y and is also a PII.

By means of the so called *equivalence theorems* (theorems A.7 and A.3 in [2]) some properties of the initial process Y are shown to be equivalent to the analogous ones of Y. The properties of Y we are dealing with are, in general, external properties on the trajectories, some of them being:

E1 : a.s. Y es lag-lad (without discontinuities of second kind) on $I \Leftrightarrow$ a.s. Y' is of limited fluctuation

E2 : a.s. Y is continuous in t (t standard) $\Leftrightarrow$ a.s. Y' is continuous in t

E3' : a.s. Y is continuous in $I \Leftrightarrow$ a.s. Y' es continuous in T and has infinitesimal increments everywhere

E4 : a.s. Y is of bounded variation in $I \Leftrightarrow$ a.s.Y' is of limited total variation in T

E4' : a.s. Y is of unbounded variation in $I \Leftrightarrow Y'$ is a.s. of unlimited total variation in T.

where in each equivalence the statement of the left is interpreted in a classical way while that of the right refers to an external property on the trajectories of Y' which holds a.s. in the non standard sense. E3' improves an equivalence stated in [2] while E4' is new. Both of them can be established by the same reasonings used in [2].

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