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MINIMIZATION OF THE FIRST EIGENVALUE IN PROBLEMS INVOLVING THE BI-LAPLACIAN

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Abstract

This paper concerns the minimization of the first eigenvalue in problems involving the bi-Laplacian under either homogeneous Navier boundary conditions or homogeneous Dirichlet boundary conditions. Physically, in case of $N = 2$, our equation models the vibration of a non homogeneous plate Ω which is either hinged or clamped along the boundary. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials inside Ω so to minimize the first mode in the vibration of the corresponding plate.

Keywords: bi-Laplacian, first eigenvalue, minimization.

Resumen

Este artículo trata de la minimización del primer autovalor en problemas relativos al bi-Laplaciano bajo condiciones de frontera homogéneas de tipo Navier o Dirichlet. Físicamente, en el problema bi-dimensional, nuestra ecuacin modela la vibración de una placa inhomogénea Ω fija con goznes a lo largo de su borde. Dados varios materiales (de diferentes densidades) y extensión total $|\Omega|$, investigamos cuál debe ser la localización de tales materiales en la placa para minimizar el primer modo de su vibración.

Palabras clave: bi-Laplaciano, primer autovalor, minimización.

Mathematics Subject Classification: 35P15, 47A75, 49K20.

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1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let g_0 be a measurable function satisfying $0 \le g_0 \le M$ in Ω, where M is a positive constant. To avoid trivial situations, we always assume $g_0 \not\equiv 0$ and $g_0 \not\equiv M$. Define $\mathcal{G}$ as the family of all measurable functions defined in Ω which are rearrangements of g_0. Consider the following eigenvalue problems

$$\Delta^2 u = \lambda g u, \quad \text{in} \quad \Omega, \qquad u = \Delta u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

and

$$\Delta^2 v = \Lambda g v, \quad \text{in} \quad \Omega, \qquad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $g \in \mathcal{G}$, $\lambda = \lambda_g$, $\Lambda = \Lambda_g$ are the first eigenvalues and u, v are the corresponding eigenfunctions. The operator Δ^2 stands for the usual bi-Laplacian, that is $\Delta^2 u = \Delta(\Delta u)$. The first eigenvalue λ of problem (1) is obtained by minimizing the associate Rayleigh quotient

$$\lambda = \inf\Big\{ \frac{\int_\Omega (\Delta z)^2 dx}{\int_\Omega g z^2 dx} : \quad z,\ \Delta z \in H_0^1(\Omega), \quad z \not\equiv 0 \Big\}. \tag{3}$$

The first eigenvalue Λ of problem (2) is obtained by minimizing the quotient

$$\Lambda = \inf\Big\{ \frac{\int_\Omega (\Delta z)^2 dx}{\int_\Omega g z^2 dx} : \quad z \in H_0^2(\Omega), \quad z \not\equiv 0 \Big\}. \tag{4}$$

It is well known that the inferior is attained in both cases [14]. The minimum of (3) satisfies problem (1) in the weak sense, that is

$$\int_\Omega \Delta u \Delta z \, dx = \lambda \int_\Omega g u z \, dx, \quad \forall z :\ z, \Delta z \in H_0^1(\Omega).$$

The minimum of (4) satisfies problem (2) in the sense

$$\int_\Omega \Delta v \Delta z \, dx = \Lambda \int_\Omega g v z \, dx, \quad \forall z \in H_0^2(\Omega).$$

By regularity results (see [1]) the solutions to problems (1) and (2) belong to $H^4_{loc}(\Omega)$.

In this paper we investigate the problems

$$\min_{g \in \mathcal{G}} \lambda_g, \qquad \text{and} \qquad \min_{g \in \mathcal{G}} \Lambda_g. \tag{5}$$

Let us give a motivation for the study of these problems in case of $N = 2$. Physically, our equations model the vibration of a non homogeneous plate Ω which is either hinged or clamped along the boundary $\partial\Omega$. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials inside Ω so to minimize the first mode in the vibration of the plate. The corresponding problem for second order equations has been discussed in several papers, see for example [6], [7], [9].

The paper is organized as follows. In Section 2 we collect some definitions and known results. In Section 3 we investigate the problems (5) proving results of existence and results of representation of minimizers. In case Ω is a ball we prove uniqueness for both problems.

2 Preliminaries

Denote with $|E|$ the Lebesgue measure of the (measurable) set $E \subset \mathbb{R}^N$. Given a measurable function $g_0(x)$ defined in Ω we say that $g(x)$, defined in Ω, belongs to the class of rearrangements $\mathcal{G} = \mathcal{G}(\}_{\prime})$ if $|\{x \in \Omega : g(x) \geq \beta\}| = |\{x \in \Omega : g_0(x) \geq \beta\}| \quad \forall \beta \in \mathbb{R}$.

We make use of the following results.

Lemma 2.1 *Let $g \in L^1(\Omega)$ and let $u \in L^1(\Omega)$. Suppose that every level set of u (that is, sets of the form $u^{-1}(\{\alpha\})$), has measure zero. Then there exists an increasing function ϕ such that $\phi(u)$ is a rearrangement of g.*

Proof. The assertion follows by Lemma 2.9 of [4]. □

Lemma 2.2 *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^r(\Omega)$, $r > 1$, $g_0 \not\equiv 0$, and let $\overline{\mathcal{G}}$ denote the weak closure of $\mathcal{G}$ in $L^r(\Omega)$. If $u \in L^s(\Omega)$, $s = r/(r-1)$, $u \not\equiv 0$, and if there is an increasing function ϕ such that $\phi(u) \in \mathcal{G}$ then*

$$\int_\Omega g\, u\, dx \leq \int_\Omega \phi(u)\, u\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\phi(u)$ is the unique maximizer relative to $\overline{\mathcal{G}}$.

Proof. The assertion follows by Lemma 2.4 of [4]. □

Lemma 2.3 *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^r(\Omega)$, $r > 1$, $g_0 \not\equiv 0$, and let $u \in L^s(\Omega)$, $s = r/(r-1)$, $u \not\equiv 0$. There exists $\overline{g} \in \mathcal{G}$ such that*

$$\int_\Omega g\, u\, dx \leq \int_\Omega \overline{g}\, u\, dx \qquad \forall g \in \overline{\mathcal{G}}.$$

Proof. It follows by Lemma 2.4 of [4]. See also [5]. □

Next we recall a well known rearrangement inequality. For u non negative in Ω, $u^\sharp$ denotes the decreasing Schwarz rearrangement of u; that is, $u^\sharp$ is defined in $\Omega^\sharp$, the ball centered in the origin with measure equal to $|\Omega|$, is radially symmetric, decreases as $|x|$ increases, and satisfies

$$|\{x \in \Omega : u(x) \geq \beta\}| = |\{x \in \Omega^\sharp : u^\sharp(x) \geq \beta\}| \quad \forall \beta \geq 0.$$

If $u \in H_0^1(\Omega)$ is non-negative and if $u^\sharp$ is the decreasing Schwarz rearrangement of u then $u^\sharp \in H_0^1(\Omega^\sharp)$ and the inequality

$$\int_{\Omega^\sharp} |\nabla u^\sharp|^2\, dx \leq \int_\Omega |\nabla u|^2\, dx \tag{6}$$

holds. The case of equality in (6) has been considered in [3]. We have

Lemma 2.4 *Let $u \in H_0^1(\Omega)$ be non-negative, and suppose equality holds in (6). If*

$$|\{x \in \Omega^\sharp :\ \nabla u^*(x) = 0,\ \ 0 < u^*(x) < \sup_\Omega u(x)\}| = 0$$

then u is a translate of $u^\sharp$.

Proof. See Theorem 1.1 of [3] or the monograph [13]. □

3 Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $M > 0$ be a real number. Let $\mathcal{G}$ be the family of all functions defined in Ω which are rearrangements of a given function g_0 with $0 \le g_0(x) \le M$, $g_0(x) \not\equiv 0$, $g_0(x) \not\equiv M$. For $g \in \mathcal{G}$, let λ_g be the first eigenvalue of problem (1), and let Λ_g be the first eigenvalue of problem (2). We investigate the problems

$$\min_{g\in\mathcal{G}} \lambda_g, \qquad \text{and} \qquad \min_{g\in\mathcal{G}} \Lambda_g.$$

Recalling (3) and (4) we can formulate the previous problems as

$$\min_{g\in\mathcal{G}} \lambda_g = \min\Big\{ \frac{\int_\Omega (\Delta z)^2 dx}{\int_\Omega g\, z^2 dx} :\ g \in \mathcal{G},\ z \in H_0^1(\Omega),\ \Delta z \in H_0^1(\Omega)\Big\}, \tag{7}$$

and

$$\min_{g\in\mathcal{G}} \Lambda_g = \min\Big\{ \frac{\int_\Omega (\Delta z)^2 dx}{\int_\Omega g\, z^2 dx} :\ g \in \mathcal{G},\ z \in H_0^2(\Omega)\Big\}. \tag{8}$$

Theorem 3.1 *Let $0 \le g_0(x) \le M$, $g_0(x) \not\equiv 0$, $g_0(x) \not\equiv M$, and let $\mathcal{G}$ be the class of all rearrangements of g_0. Then*
a) *there exists $\overline{g} \in \mathcal{G}$ such that*

$$\lambda_{\overline{g}} = \min_{g\in\mathcal{G}} \lambda_g;$$

b) *there exists $\tilde{g} \in \mathcal{G}$ such that*

$$\Lambda_{\tilde{g}} = \min_{g\in\mathcal{G}} \Lambda_g.$$

Proof. We prove first part a). Let

$$I\ = \inf_{g\in\mathcal{G}} \lambda_g = \lim_{i\to\infty} \lambda_{g_i} = \lim_{i\to\infty} \frac{\int_\Omega (\Delta u_i)^2 dx}{\int_\Omega g_i u_i^2 dx}, \tag{9}$$

where $u_i = u_{g_i}$ is the eigenfunction corresponding to g_i normalized so that

$$\int_\Omega u_i^2 dx = 1.$$

We may assume that the sequence $\{\lambda_{g_i}\}$ is decreasing. By (9) and the latter equation we get

$$\int_\Omega (\Delta u_i)^2 dx \le \lambda_{g_1} M. \tag{10}$$

On the other side, since u_i vanishes on $\partial\Omega$, by Lemma 9.17 of [11] we have

$$\|u_i\|_{H^2(\Omega)} \le C \|\Delta u_i\|_{L^2(\Omega)}$$

with C independent of i. It follows that the norms $\|u_i\|_{H^2(\Omega)}$ and $\|\Delta u_i\|_{L^2(\Omega)}$ are equivalent. This fact and (10) imply that the sequence $\{u_i\}$ is bounded in the $H^2(\Omega)$ norm and some subsequence (still denoted $\{u_i\}$) converges weakly in $H^2(\Omega)$ to a function $\overline{u}$. We can also assume that $\{u_i\}$ converges strongly to $\overline{u}$ in $L^{2+\epsilon}(\Omega)$ for some $\epsilon > 0$. Furthermore, since $\{g_i\}$ is bounded in $L^\infty(\Omega)$, it must contain a subsequence (still denoted $\{g_i\}$) converging weakly to $\eta \in L^r(\Omega)$ for any $r > 1$. We have

$$\int_\Omega g_i u_i^2 dx - \int_\Omega \eta \overline{u}^2 dx = \int_\Omega (g_i - \eta)\overline{u}^2 dx + \int_\Omega g_i(u_i^2 - \overline{u}^2) dx.$$

We find

$$\lim_{i\to\infty} \int_\Omega (g_i - \eta)\overline{u}^2 dx = 0,$$

because $\overline{u}^2 \in L^s(\Omega)$ for some $s > 1$ and $g_i \to \eta$ weakly in $L^r(\Omega)$ for $r = s/(s-1)$. Moreover,

$$\lim_{i\to\infty} \int_\Omega g_i(u_i^2 - \overline{u}^2) dx = 0.$$

The latter result can be proved by using Lebesgue's theorem as follows. Since $u_i \to \overline{u}$ in $L^2(\Omega)$, we have (up to a subsequence)

$$\lim_{i\to\infty} g_i(u_i^2 - \overline{u}^2) = 0 \ \text{ a.e. in } \ \Omega,$$

and

$$g_i|u_i^2 - \overline{u}^2| \le M(\psi^2 + \overline{u}^2),$$

for some integrable function ψ^2. Indeed, since u_i converges in $L^2(\Omega)$ one can find $\psi \in L^2(\Omega)$ such that $u_i(x) \le \psi(x)$ *a.e.* for some subsequence of u_i [10]. Hence,

$$\lim_{i\to\infty} \int_\Omega g_i u_i^2 dx = \int_\Omega \eta\, \overline{u}^2 dx. \tag{11}$$

By Lemma 2.3 we can find $\overline{g} \in \mathcal{G}$ such that

$$\int_\Omega \eta\, \overline{u}^2 dx \le \int_\Omega \overline{g}\, \overline{u}^2 dx. \tag{12}$$

On the other side, from the inequality

$$0 \le \int_\Omega \big(\Delta(u_i - \overline{u})\big)^2 dx = \int_\Omega (\Delta u_i)^2 dx - 2\int_\Omega \Delta u_i \Delta \overline{u}\, dx + \int_\Omega (\Delta \overline{u})^2 dx$$

and the weak convergence of $\{u_i\}$ to $\overline{u}$ in $H^2(\Omega)$ we find

$$\liminf_{i\to\infty} \int_\Omega (\Delta u_i)^2 dx \ge \int_\Omega (\Delta \overline{u})^2 dx.$$

By using the latter result together with (11) and (12) we have

$$I = \lim_{i\to\infty} \frac{\int_\Omega (\Delta u_i)^2 dx}{\int_\Omega g_i u_i^2 dx} \ge \frac{\int_\Omega (\Delta \overline{u})^2 dx}{\int_\Omega \eta\, \overline{u}^2 dx} \ge \frac{\int_\Omega (\Delta \overline{u})^2 dx}{\int_\Omega \overline{g}\, \overline{u}^2 dx}. \tag{13}$$

Our minimizing sequence u_i satisfies (in a weak sense)

$$\Delta(\Delta u_i) = \lambda_{g_i} g_i u_i, \quad \Delta u_i \in H_0^1(\Omega).$$

If we multiply by $-\Delta u_i$ and integrate over Ω, after simplification we find

$$\|\nabla(\Delta u_i)\|_{L^2(\Omega)} \le \lambda_{g_i} \|g_i u_i\|_{L^2(\Omega)}.$$

Since λ_{g_i} is decreasing, $0 \le g_i \le M$ and $\|u_i\|_{L^2(\Omega)} = 1$ we find that $\|\nabla(\Delta u_i)\|_{L^2(\Omega)} \le \lambda_{g_1} M$. As a consequence, since $\Delta u_i \in H_0^1(\Omega)$ we also have $\Delta \overline{u} \in H_0^1(\Omega)$. Therefore, if $\lambda_{\overline{g}}$ is the (first) eigenvalue corresponding to $\overline{g}$ in problem (1), and if $u_{\overline{g}}$ is a corresponding eigenfunction then by (3) we have

$$\frac{\int_\Omega (\Delta \overline{u})^2 dx}{\int_\Omega \overline{g}\, \overline{u}^2 dx} \ge \frac{\int_\Omega (\Delta u_{\overline{g}})^2 dx}{\int_\Omega \overline{g}\, u_{\overline{g}}^2 dx} = \lambda_{\overline{g}} \ge I.$$

By the latter result and (13) we must have $I = \lambda_{\overline{g}}$. Part a) of the theorem is proved.

The proof of part b) is similar. Define

$$\tilde{I} = \inf_{g \in \mathcal{G}} \Lambda_g = \lim_{i \to \infty} \frac{\int_\Omega (\Delta v_i)^2 dx}{\int_\Omega g_i v_i^2 dx},$$

where $v_i = v_{g_i}$ is the eigenfunction corresponding to g_i normalized so that

$$\int_\Omega v_i^2 dx = 1.$$

Of course, $\{g_i\}$ is not, in general, the same as for part a). Arguing as in the previous case we find that v_i is bounded in the norm of $H^2(\Omega)$. Therefore, a subsequence (still denoted $\{v_i\}$) converges weakly in $H^2(\Omega)$ to a function $\tilde{v} \in H_0^2(\Omega)$. We can also assume that $\{v_i\}$ converges strongly to $\tilde{v}$ in $L^{2+\epsilon}(\Omega)$ for some $\epsilon > 0$. Furthermore, $\{g_i\}$ must contain a subsequence (still denoted $\{g_i\}$) converging weakly to some $\zeta \in L^r(\Omega)$ for any $r > 1$. Hence,

$$\lim_{i \to \infty} \int_\Omega g_i v_i^2 dx = \int_\Omega \zeta\, \tilde{v}^2 dx.$$

By Lemma 2.3 we can find $\tilde{g} \in \mathcal{G}$ such that

$$\int_\Omega \zeta\, \tilde{v}^2 dx \le \int_\Omega \tilde{g}\, \tilde{v}^2 dx.$$

Moreover we have

$$\liminf_{i \to \infty} \int_\Omega (\Delta v_i)^2 dx \ge \int_\Omega (\Delta \tilde{v})^2 dx.$$

Using the last three results we find

$$\tilde{I} = \lim_{i \to \infty} \frac{\int_\Omega (\Delta v_i)^2 dx}{\int_\Omega g_i v_i^2 dx} \ge \frac{\int_\Omega (\Delta \tilde{v})^2 dx}{\int_\Omega \zeta\, \tilde{v}^2 dx} \ge \frac{\int_\Omega (\Delta \tilde{v})^2 dx}{\int_\Omega \tilde{g}\, \tilde{v}^2 dx}. \tag{14}$$

Recall that $\tilde{v} \in H_0^2(\Omega)$ and $\tilde{g} \in \mathcal{G}$. If $\Lambda_{\tilde{g}}$ is the (first) eigenvalue corresponding to $\tilde{g}$ in problem (2), and if $v_{\tilde{g}}$ is a corresponding eigenfunction then, using (4) we have

$$\frac{\int_\Omega (\Delta \tilde{v})^2 dx}{\int_\Omega \tilde{g}\, \tilde{v}^2 dx} \geq \frac{\int_\Omega (\Delta v_{\tilde{g}})^2 dx}{\int_\Omega \tilde{g}\, v_{\tilde{g}}^2 dx} = \Lambda_{\tilde{g}} \geq \tilde{I}. \tag{15}$$

By (14) and (15) we must have $\tilde{I} = \Lambda_{\tilde{g}}$. The theorem is proved. □

We prove the so called Euler-Lagrange equation for solutions of our minimization problems. Actually, there is a difference between the two cases. Concerning problem (1), we know that the first eigenfunction does not change sign, and we can assume that it is positive in Ω. Concerning problem (2), there are domains Ω such that the corresponding first eigenfunction is sign changing, and there are domains such that the corresponding first eigenfunction is positive: see [12] and references therein.

In what follows we write $\{g(x) > 0\}$ instead of $\{x \in \Omega : g(x) > 0\}$.

Theorem 3.2 *a) Suppose $\overline{g}$ is a solution to problem* (7). *There exists an increasing function ϕ such that*

$$\overline{g} = \phi(u_{\overline{g}}).$$

b) Suppose $\overline{g}$ is a solution to problem (8) *and that Ω is such that the corresponding first eigenfunction of problem* (2) *is positive. There exists an increasing function φ such that*

$$\overline{g} = \varphi(u_{\overline{g}}).$$

Proof. If $u_{\overline{g}}$ is the positive normalized eigenfunction corresponding to the minimizer $\overline{g}$ of problem (7), for any $g \in \mathcal{G}$ we have

$$\frac{\int_\Omega (\Delta u_{\overline{g}})^2 dx}{\int_\Omega \overline{g} u_{\overline{g}}^2 dx} \leq \frac{\int_\Omega (\Delta u_{\overline{g}})^2 dx}{\int_\Omega g u_{\overline{g}}^2 dx}.$$

Hence,

$$\int_\Omega g u_{\overline{g}}^2\, dx \leq \int_\Omega \overline{g} u_{\overline{g}}^2\, dx \tag{16}$$

for all $g \in \mathcal{G}$.

On the other side, we know that the function $u_{\overline{g}}$ satisfies the eigenvalue equation

$$\Delta^2 u_{\overline{g}} = \lambda\, \overline{g} u_{\overline{g}}.$$

If $-\Delta u_{\overline{g}} = v$, by the above equation we have $-\Delta v \geq 0$ in Ω and $v = 0$ on $\partial\Omega$. It follows that $v(x) > 0$ in Ω. Since $-\Delta u_{\overline{g}} > 0$, the function $u_{\overline{g}}$ cannot have level sets of positive measure. Hence, by Lemma 2.1, inequality (16) and Lemma 2.2 we infer the existence of an increasing function ϕ_1 such that $\overline{g} = \phi_1(u_{\overline{g}}^2)$. Thus, part a) of the theorem follows with $\phi(t) = \phi_1(t^2)$.

If $u_{\overline{g}}$ is the positive normalized eigenfunction corresponding to the minimizer $\overline{g}$ of problem (8), inequality (16) holds for all $g \in \mathcal{G}$. Moreover, $\Delta^2 u_{\overline{g}} = \Lambda\, \overline{g} u_{\overline{g}}$. By this equation, the function $u_{\overline{g}}$ cannot have level sets of positive measure on $\{\overline{g}(x) > 0\}$. If the set $\{\overline{g}(x)=0\}$ has zero measure, by Lemma 2.1, inequality (16) and Lemma 2.2 we infer the existence of an increasing function φ_1 such that $\overline{g}=\varphi_1(u_{\overline{g}}^2)$. Thus, in this case part b) of the theorem follows with $\varphi(t)=\varphi_1(t^2)$. Otherwise, setting $E=\{\overline{g}(x)=0\}$, we define:

$$S=\sup_{x\in E}(u_{\overline{g}}(x))^2.$$

By using (16) one proves that $(u_{\overline{g}}(x))^2\geq S$ on $\{\overline{g}(x)>0\}$ a.e. For the proof of this result we refer to [8], Theorem 3.2. Since $u_{\overline{g}}$ cannot have level sets of positive measure on $\Omega\setminus E$, by Lemma 2.1 we infer the existence of an increasing function $\varphi_1:(S,\infty)\to[0,M]$ such that $\varphi_1(u_{\overline{g}}^2)$ is a rearrangement of $\overline{g}$ on $\Omega\setminus E$. Now we define an increasing function φ_2 as

$$\varphi_2(t)=\begin{cases}0 & t\leq S\\ \varphi_1(t) & t>S.\end{cases}$$

Since $\varphi_2(u_{\overline{g}}^2)$ is a rearrangement of $\overline{g}$ on Ω, by inequality (16) and Lemma 2.2 we infer that $\overline{g}=\phi_2(u_{\overline{g}}^2)$. Part b) of the theorem follows taking $\varphi(t)=\varphi_2(t^2)$. The theorem is proved. $\square$

Remarks. Theorem 3.2 gives some information on the location of the materials in order to minimize the first eigenvalue of problem (7). Indeed, since the associate eigenfunction $u_{\overline{g}}$ vanishes on the boundary $\partial\Omega$, and $\overline{g}=\phi(u_{\overline{g}})$ with ϕ increasing, the material with higher density must be located where $u_{\overline{g}}$ is large, that is, far from $\partial\Omega$. The same remark holds for problem (8) in appropriate domains.

Theorem 3.3 *Let B be a ball in $\mathbb{R}^N$, and let g be a minimizer of either problem* (7) *or problem* (8) *with $\Omega=B$. Then $g=g^\sharp$.*

Proof. If g is a minimizer of problem (7) and if $u=u_g$ is a corresponding positive eigenfunction we have

$$\lambda_g=\frac{\int_B(\Delta u)^2dx}{\int_B gu^2dx}. \tag{17}$$

Put

$$-\Delta u=z. \tag{18}$$

Then

$$-\Delta z=\lambda_g gu.$$

Since $u>0$ in B and $z=0$ on ∂B we have $z>0$ in B. If $z^\sharp$ is the Schwarz decreasing rearrangement of z then $z^\sharp\in H_0^1(B)$ and

$$\int_B(\Delta u)^2dx=\int_B(z^\sharp)^2dx. \tag{19}$$

Furthermore, if $\overline{u}$ is the solution to the problem

$$-\Delta\overline{u}=z^\sharp \ \text{ in } \ B,\ \ \overline{u}=0 \ \text{ on } \ \partial B \tag{20}$$

then, by a result of G. Talenti ([15], Theorem 1, (iv)) we have

$$u^\sharp \le \overline{u} \ \text{ in } \ B. \tag{21}$$

By a well known inequality on rearrangements and (21) we find

$$\int_B g u^2 dx \le \int_B g^\sharp (u^\sharp)^2 dx \le \int_B g^\sharp (\overline{u})^2 dx. \tag{22}$$

Since $(z^\sharp)^2 = (\Delta \overline{u})^2$, by (19), (17) and (22) we find

$$\lambda_g \ge \frac{\int_B (\Delta \overline{u})^2 dx}{\int_B g^\sharp (\overline{u})^2 dx} \ge \frac{\int_B (\Delta u_{g^\sharp})^2 dx}{\int_B g^\sharp (u_{g^\sharp})^2 dx} = \lambda_{g^\sharp}.$$

In the last step we have used the fact that $\overline{u}$ is admissible (because $\overline{u} = \Delta \overline{u} = 0$ on ∂B) and the variational characterization of $\lambda_{g^\sharp}$. Since λ_g is a minimizer, we must have $\lambda_{g^\sharp} = \lambda_g$ and equality must hold in (22). In particular,

$$\int_B g^\sharp (u^\sharp)^2 dx = \int_B g^\sharp (\overline{u})^2 dx.$$

Recalling that $g(x)$ is positive in a set of positive measure we have $g^\sharp(x) > 0$ in a ball $B(r_0)$ of radius r_0 for some $r_0 > 0$. Therefore the previous equation and (21) imply that $u^\sharp(0) = \overline{u}(0)$. An inspection of the proof of Talenti's result [15] (see also [2]) yields $u^\sharp(x) = \overline{u}(x)$ in all of B. Moreover by (18), (20) with $\overline{u} = u^\sharp$, and (6) we find

$$\int_B |\nabla u|^2 dx = \int_B uz dx \le \int_B u^\sharp z^\sharp dx = \int_B |\nabla u^\sharp|^2 dx \le \int_B |\nabla u|^2 dx.$$

It follows that

$$\int_B |\nabla u|^2 dx = \int_B |\nabla u^\sharp|^2 dx.$$

By Lemma 2.4 we get $u(x) = u^\sharp(x)$ in B. Furthermore, by Theorem 3.2 a) we have $g = \phi(u)$ for some increasing function ϕ. This implies that g is radially symmetric and decreasing, hence $g = g^\sharp$. The theorem is proved in this case.

Let us come to problem (8). Putting $-\Delta v = w$ and recalling that $\frac{\partial v}{\partial \nu} = 0$ on ∂B we find

$$\int_B w \, dx = -\int_B \Delta v \, dx = \int_{\partial B} \frac{\partial v}{\partial \nu} d\sigma = 0.$$

This means that $w(x)$ is sign changing in B. Let $w^\sharp(x)$ be the signed Schwarz decreasing rearrangement of $w(x)$ and let

$$-\Delta \overline{v} = w^\sharp \ \text{ in } \ B, \ \ \overline{v} = 0 \ \text{ on } \ \partial B.$$

Since $\int_B w^\sharp dx = 0$ the result of Talenti [15] continues to hold as observed also in [2]. Hence,

$$v^\sharp \le \overline{v} \ \text{ in } \ B.$$

Moreover, since

$$0 = -\int_B w \, dx = -\int_B w^\sharp dx = \int_B \Delta \overline{v} \, dx = \int_{\partial B} \frac{\partial \overline{v}}{\partial \nu} d\sigma = \frac{\partial \overline{v}}{\partial \nu} |\partial B|,$$

we have $\overline{v} \in H_0^2(B)$. The proof continues as in the previous case. □

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