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Delamotte Approximation

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Abstract: We solve the anharmonic potential with an approximate, non perturbative method of B. Delamotte. The numerical problem is analyzed and the analytical solution is obtained in Jacobi Elliptical functions. We show, that the Fourier expansion of $Sd(u||m)$ the Jacobi function is the Delamotte approximation. We analyze the problems of the method when there are several equilibrium points. As in Duffing equation. For the limit cycles we use the simple nonlinear oscillation as a model to study the convergency of the method. The problems about the uniqueness of the approximation are studied. A theorem is proved that gives a practical criteria when to use the method for Hamiltonian systems.

Subject headings: Anharmonic potential, Fourier expansion, Jacobi function, limit cycles, Duffing equation

Resumen: Se resuelve el potencial anarmónico con un método de aproximación no perturbativo. El problema numérico es analizado y la solución analítica se obtiene en función de funciones elípticas de Jacobi. Mostramos que la expansión de Fourier de $Sd(u||m)$ es la función de Jacobi y a su vez es la aproximación de Delamotte. Analizamos los problemas del método cuando hay varios puntos de equilibrio, similar a los de la ecuación de Duffing. Para los ciclos límites usamos un modelo nolineal de oscilación para estudiar la convergencia del método. Los problemas de unicidad de la aproximación son también estudiados. Se prueba un teorema que da un criterio práctico cuando se utiliza el método para sistemas hamiltonianos.

Encabezados de materia: Potential anarmónico, expansion de Fourier, función de Jacobi, ciclo límite, ecuación de Duffing

1. Introducction

Dr. B. Delamotte of the Laboratory of High Energy Physics from Paris University, published (Delamotte, 1993) what we call the Delamotte ansatz. The ansatz is a non perturbative method for solving second order differential equations. The method is specially good for periodic solutions with only one critical point and the solution it's obtained independent of the values of the "perturbation parameter", but our work shows that when the differential equation is studied as dependent of a second parameter for the critical points the same method that works for some values of the parameter does not work for others. This essentially means that, as the Fourier spectrum of the solution become broad the convergence of the method is slow. In other words as the first Fourier coefficients are small, the slower the convergence is. Limitations in our Mathematica (Wolfram Research) prevented us of studying the convergence in further detail.

Delamotte ansatz is the following †. Let us call x(t) the solution of the second order differential autonomous equation:

$$f(x(t), x'(t), x''(t)) = 0 \qquad (I-1)$$

subject to the initial conditions

$$x(t_0) = x_0, \qquad x'(t_0) = x'_0 \qquad (I-2)$$

The principle of the method is to replace eq.(I.1) and (I.2) by a linear differential equation with a explicit time dependent right-hand side. In the language of classical

† The equations I.n are from Delamotte article, 1993.

mechanics an external force, that forces the harmonic potential to follow the trajectory x(t). This force always exists and is given by:

$$x''(t) + \omega^2 x(t) = F(t) \qquad (I-3)$$

where ω is a free parameter. Note that F depends on the particular differential equation. The principle of the method is to make the ansatz:

$$F_{ans}(t) = x''_{ans} + \omega^2 x_{ans}(t) \qquad (I.4)$$

if δx is the difference between x and x_{ans}

$$x(t) = x_{ans}(t) + \delta x(t) \qquad (I.5)$$

and x_{ans} is close enough to x if:

$$|\delta x| \ll |x_{ans}|; \qquad |\delta x'| \ll |x'_{ans}|; \qquad |\delta x''| \ll |x''_{ans}|. \qquad (I.6)$$

In practice, x_{ans} is expanded on a basis of functions, in our case as a Fourier sum:

$$x_{ans}(t) = \sum_{k=0}^{N} x_k sin(k\omega t) + y_k cos(k\omega t). \qquad (I.7)$$

2. The anharmonic potential

Let us consider the anharmonic oscillator of equation:

$$x'' = -\omega_0^2 x - g x^3 \qquad (I.8)$$

$$\delta x'' = -\omega_0^2 \delta x - (\omega_0^2 - \omega^2) x_{ans} - g(x_{ans} + \delta x)^3 - F_{ans} \qquad (I.9)$$

using I.6 we have:

$$(\omega_0^2 - \omega^2) x_{ans} + g x_{ans}^3 + F_{ans} \sim 0 \qquad (I.10)$$

Let V(x) be the anharmonic potential:

$$V(x) = \frac{1}{2} x^2 + \frac{g}{4} x^4 \qquad (1)$$

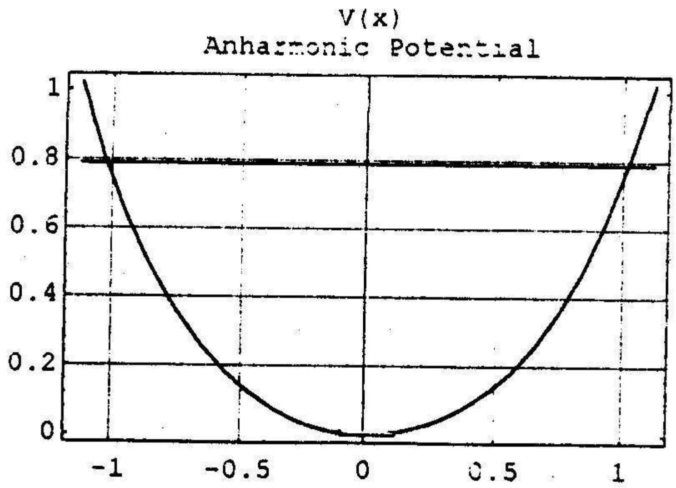

Fig. 1 The anharmonic potential and total energy

the corresponding differential equation is:

$$\frac{d^2x}{dt^2} + x + gx^3 = 0 \tag{2}$$

where we did the mass m, the spring constant k and the frequency w_0^2 equal to 1. The energy conservation could be written as:

$$\frac{1}{2}v^2 + \frac{1}{2}x^2 + \frac{g}{4}x^4 = \frac{1}{2}v_0^2 \tag{3}$$

The turning point has equation:

$$gx^4 + 2x^2 - 2v_0^2 = 0 \tag{4}$$

The solution of equation (4) is:

$$x_r = \pm\left\{\frac{[1 + 2gv_0^2]^{1/2} - 1}{g}\right\}^{1/2} \tag{5}$$

2.1. The First Integral

The energy or first integral of equation (2) is:

$$\int_0^x \frac{dz}{\sqrt{2v_0^2 - 2z^2 - gz^4}} \tag{6}$$

Equation (6) could be integrated (Abramowitz and Stegun, 1972). In order to use the formula we have to complete the square.

$$2v_0^2 - 2x^2 - gx^4 = g\left\{\sqrt{\frac{2v_0^2}{g}+\frac{1}{g^2}}-(\frac{1}{g}+x^2)\right\}\left\{\sqrt{\frac{2v_0^2}{g}+\frac{1}{g^2}}+(\frac{1}{g}+x^2)\right\} \quad (7)$$

This means that:

$$a^2 = \sqrt{\frac{2v_0^2 g+1}{g^2}}+\frac{1}{g} \quad (8)$$

$$b^2 = \sqrt{\frac{2v_0^2 g+1}{g^2}}-\frac{1}{g} \quad (9)$$

The solution which we call Se(t) is:

$$Se(t) = \frac{v_0}{[1+2v_0^2 g]^{1/4}} Sd\left([1+2v_0^2 g]^{1/4} t || \frac{1}{2}(1-\frac{g^2}{\sqrt{1+2v_0^2 g}}\right) \quad (10-A)$$

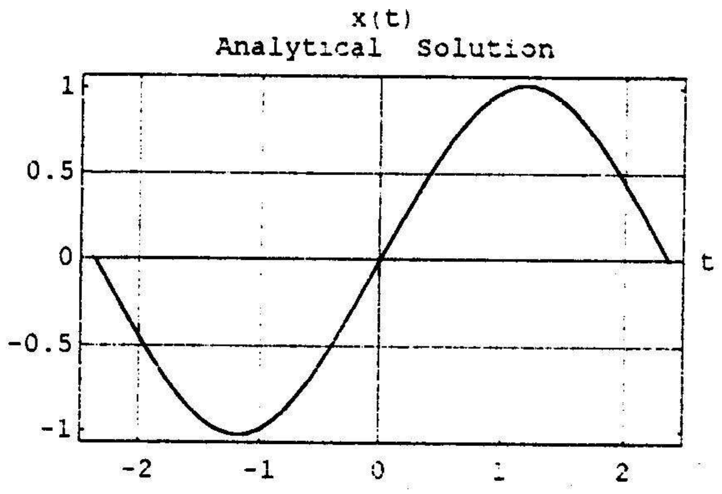

Fig. 2 The solution of the anharmonic potential

This is the analytical solution of the x^4 potential for a particle with $x(0) = 0$ and $v(0) = v_0$. Results are shown in the accompanying Mathematica Notebook for several values of g and the initial condition v_0.

3. The Generalization to Duffing Equation

The equation for the anharmonic oscillator that we have analyzed could be generalized by the introduction of a parameter λ, the equation could be written as (Hale and Kocak, 1991):

$$\frac{d^2w}{dt^2} + \lambda w + gw^3 = 0 \tag{11}$$

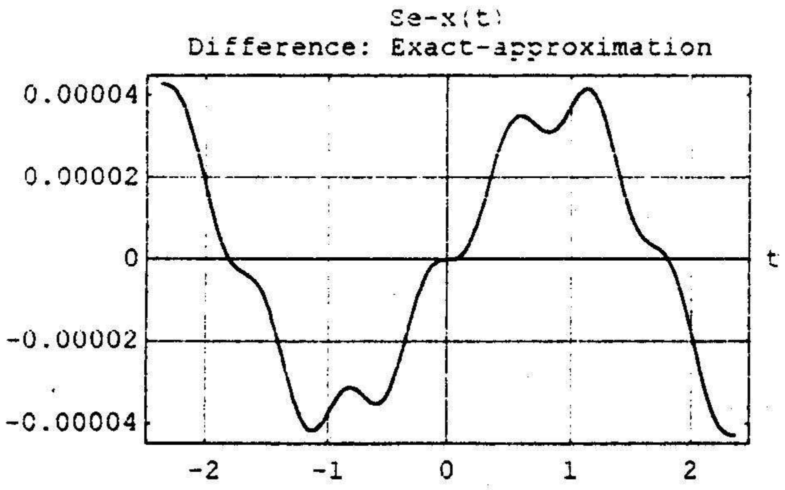

Fig. 3 Difference between analytical and Delamotte approximatic

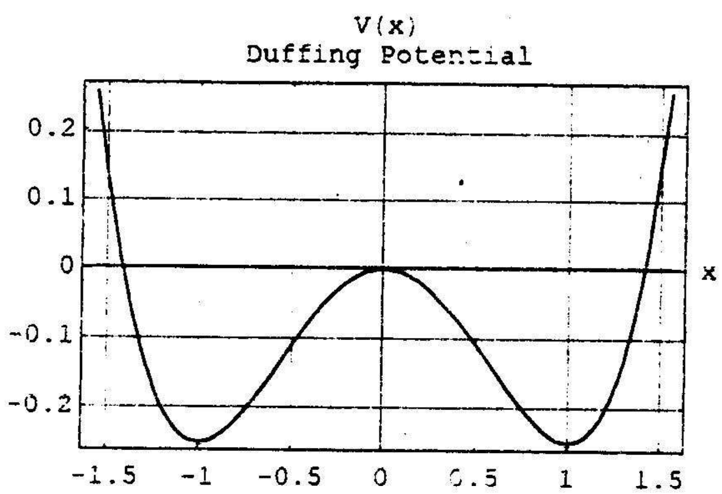

Fig. 4 Duffing potential

where now $\lambda \epsilon [-1,1]$ is a parameter for the sign of linear force, besides g, that is the parameter of the strength of the perturbation.

For figures 4, 5 and 6 we have chosen:

for equation 11 $\lambda = -1 \, and \, g = 1$.

for the inital conditions I.2, $x_0 = 0 \, and \, v_0 = -0.0467$.

The analytical solution of equation 11 is a modification of equation 10-A:

$$Se(t) = \frac{v_0 \lambda^{1/4}}{[\lambda + 2v_0^2 g]^{1/4}} Sd\left([\lambda(\lambda + 2v_0^2 g)]^{1/4} t || \frac{1}{2}(1 - \frac{g^2}{\lambda^{3/2}\sqrt{\lambda + 2v_0^2 g}}\right) \qquad (10-B)$$

Equation 11 could be written as a linear system with the substitution:

$$x = w \qquad y = \frac{dw}{dt} \qquad (12)$$

Then

$$\frac{dx}{dt} = y = f(x,y)$$

$$\frac{dy}{dt} = -\lambda x - gx^3 = g(x,y) \qquad (13)$$

We now look for the equilibrium points given by the equations:

$$f(x,y) = 0 \qquad g(x,y) = 0 \qquad (14)$$

with solutions:

$$(0,0) \qquad (\pm\sqrt{-\frac{\lambda}{g}}, 0) \qquad (15)$$

If $\lambda \geq 0$ there is only 1 equilibrium point the origin (0,0), if $\lambda \leq 0$ there are 3 equilibrium points given by equation 15. We need to relate the stability of the equilibrium points with the application of the Delamotte ansatz, thus we study the stability of the 3 equilibrium points through the linearization. Let's write the Floquet matrix:

$$\begin{pmatrix} \frac{\partial f(x_0,y_0)}{\partial x} & \frac{\partial f(x_0,y_0)}{\partial y} \\ \frac{\partial g(x_0,y_0)}{\partial x} & \frac{\partial g(x_0,y_0)}{\partial y} \end{pmatrix}$$

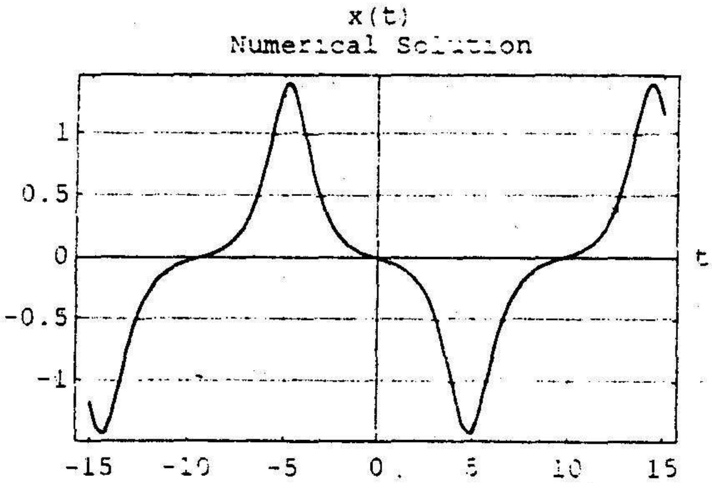

Fig. 5 Duffing numerial solution

Using definition 13 for f and g we obtain:

$$m = \begin{pmatrix} 0 & 1 \\ -\lambda - 3gx^2 & 0 \end{pmatrix} \tag{16}$$

In order to study the stability of each equilibrium point we need the eigenvalues of m in equation 16 for each point.

$$m_{\pm} = \pm\sqrt{-\lambda - 3gx^2} \tag{17}$$

now we have 3 cases to analyse:

1) (0,0) $m_{\pm} = \pm\sqrt{-\lambda}$ if $\lambda < 0$, $m_{\pm} = \pm\sqrt{|\lambda|}$ then it has 2 eigenvalues different from 0 and opposite signs and (0,0) is a saddle.

2) In the case $\lambda < 0$ and $x = \pm\sqrt{\frac{-\lambda}{g}}$ the eigenvalues are $m_{\pm} = \pm i\sqrt{2|\lambda|}$ the result is a center we can not apply the theorem to the linearization but the potential theory say it's a minimum or stable equilibrium point (Hale and Kocak, 1991). For this case the Delamotte ansatz does not work, there are several real roots for the values of ω and x_1 etc.

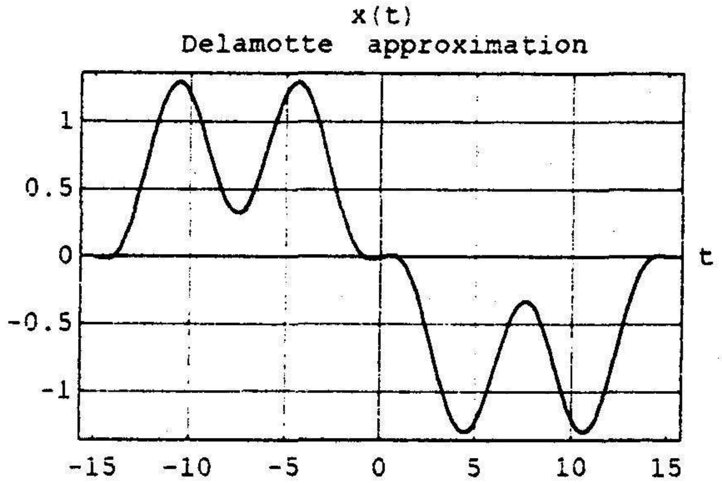

Fig. 6 Order 2 Delamotte approximation to Duffing

4. Numerical Conclusions

Our notebooks on Mathematica show (but not prove):

1) Convergency of Delamotte ansatz to the Fourier coefficients in the anharmonic potential.
2) In the case $\lambda < 0$, the method works for a big total energy E.
3) But as $E \rightarrow 0$ or $E < 0$ there are inconsistencies not only related to Delamotte ansatz but to the analytical method of solution. In the case of Delamotte ansatz several real roots appear in (I.10), in the analytical case you have to choose a proper analytical extension for equation (10-B). The real problem is near the bifurcation $E \rightarrow 0$, because once you have chosen a specific "vacuum" (minimum of the potential) the solution could be found in the neighborhood. In the following section we state some conclusions of our numerical work under the generic name of Delamotte theorems.

5. Delamotte Theorems

As we have seen the Delamotte ansatz gives a very good approximation for some kinds of differential equations. The big problems are to characterize the specific class of differential equations and to prove the convergency to the Fourier expansion of the solution. We write some theorems, which give specifics implementations of Delamotte ansatz. We divide the theorems in 2 cases:

case 1: the Hamiltonian systems with a unique minimum of the potential. As we have seen these systems oscillate and the Fourier series expansion is natural (Elsgoltz, 1975).

case 2: dissipative systems with a unique limit cycle.

5.1. Hamiltonian Systems

A Hamiltonian system is one with a constant Hamiltonian, the energy, for this system a potential V(x) exists and the force is written as:

$$F = -\frac{\partial V(x)}{\partial x} \tag{18}$$

then the equation I.1 is written as:

$$\frac{\partial x}{\partial t^2} + \frac{\partial V}{x} = 0 \qquad x(0) = x_0, \qquad x'(0) = v_0 \tag{19}$$

and the energy E:

$$E = \frac{1}{2}(\frac{dx}{dx})^2 + V(x) \tag{20}$$

is a constant.

Theorem # 1

Let be $V(x)\epsilon C^2[a,b]$ and $x(t)\epsilon C^2[0,T]$ where T is the period of the solution and the differential equation:

$$\frac{\partial x}{\partial t^2} + \frac{\partial V}{x} = 0 \qquad x(0) = x_0, \qquad x'(0) = v_0 \tag{19}$$

if $V(x) = E$ has only two real roots a,b and only one local minimum of V(x), $a < x_{min} < b$; then, exist:

A driving force; where w is a free parameter

$$F_{ans}(t) = x''_{ans} + \omega^2 x_{ans}(t) \qquad (I.4)$$

An expansion,

$$x_{ans}(t) = \sum_{k=1}^{N} x_k sin(k\omega t) + y_k cos(k\omega t) \qquad (I.7)$$

And if x_{ans} is close enough,

$$|\delta x| \ll |x_{ans}|; \qquad |\delta x'| \ll |x'_{ans}|; \qquad |\delta x''| \ll |x''_{ans}| \qquad (I.6)$$

I-7 is the Fourier series of the solution $x(t)$.

Proof

Assume that equation 19 satisfy the conditions of the theorem of existence, uniqueness and smoothness (Hale and Kocak, 1991) and that the interval [0,T] is contained on the maximal interval of existence of the equation. We define the Delamotte topology in the sense that if $g, f \epsilon C^2[0,T]$

$$||g|| = \sup_{t\epsilon[0,T]} |g(t)|$$

There is a technical term for Delamotte topology is called the C^2 topology (Hale and Kocak, 1991):

$$||f - g||_2 = \sup ||D^i f(x) - D^i g(x)||_{i\leq 2}$$

in other words Delamotte convergence:

$$||g - f|| \ll ||f||$$

is convergence in the uniform norm:

$$\sup_{t\epsilon[0,T]} ||g(t) - f(t)|| < \epsilon \sup_{t\epsilon[0,T]} |f(t)|$$

as we know the uniform norm dominates the L^2 semi norm (Lang, 1975) and together with the theorem (Lang, 1975) about the uniqueness of the Fourier expansion, the Delamotte expansion is the Fourier expansion.

See that the other conditions about the derivatives are necessary to obtain the coefficients and the condition about the roots are in order to have a fast numerical convergence. For example the second difference (the last term of equation I-6) essentially say that the convergence is faster that $\frac{C}{n^2}$. The limitation in V(x) is not essential for the convergence but gives to the method a practical condition when the convergence is fast.

6. Limit Cycles

6.1. Lienard differential equation

In this section we use the simple nonlinear oscillations (Smith, 1961) in order to study the value of Delamotte approximation to differential equations with a unique limit cycle.(Hale and Kocak, 1991). The Smith equations are worth to mention because they have a formal solution in terms of elementary functions, the associated limit cycles are algebraic curves and for some values of the parameters the differential equation is similar to Van der Pol equation. The Lienard differential equation is:

$$x''(t) + x'(t)f(x) + g(x) = 0 \tag{21}$$

It is know that equation 21 has a unique stable periodic solution if:

f(x) is continuous and even, and $f(0) < 0$

g(x) is continuous, satisfies the Lipschitz condition, and $xg(x) > 0$, for $x \neq 0$

$F(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$, respectively, where $F(x) = \int_0^x f(u)du$, and $F(x)$ has a single positive zero at $x = a$, while for $x \geq a$, $F(x)$ increases monotonically. We make the Smith choice:

$$f(x) = (n+2)bx^n - 2a \qquad g(x) = x[c + (bx^n - a)^2] \tag{22}$$

and a,b,$c = \omega^2$ and n are constants. The general solution is:

7. Referencias

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$$x(t) = \frac{cos(p+\omega t)}{\left\{qe^{-nat} + nbe^{-nat}\int_0^t e^{na\theta}cos^n(p+\omega\theta)d\theta\right\}^{1/n}} \tag{23}$$

We shall study the special case when a=2, b=1, c=1 and n=2, in this case equation 21 is similar to Van der Pol equation:

$$x'' + 4x'[x^2-1] + x + x(x^2-2)^2 = 0 \tag{25}$$

The general solution to this equation is:

$$x(t) = \frac{\cos(p+t)}{\left\{qe^{-4t} + \frac{1}{4} + \frac{1}{10}[2\cos(2(p+t)) + \sin(2(p+t))]\right\}^{1/2}} \tag{26}$$

where p and q are arbitrary constants. When q=0 the solution is periodic, and is easy to see that the period is $T = 2\pi$ in this case we wrote equation 26 as:

$$x(t) = \frac{\cos(p+t)}{\sqrt{\frac{1}{4} + \frac{1}{10}[2\cos(2(p+t)) + \sin(2(p+t))]}} \tag{27}$$

from equation 27 we obtain the special initial conditions for the limit cycle in the case $p = \pm\frac{\pi}{2}$.This is the case when initially the particle is in the origin and has an initial velocity v_0, since q and p are fixed $v_0 = \mp 2\sqrt{5}$ in other words the special initial conditions for the limit cycle we choose are:

$$x(0) = x_0 = 0 \qquad x'(0) = v_0 = \mp 2\sqrt{5} \tag{28}$$

this solution has period $T = 2\pi$ and amplitude $x_{max} = 2.0$.

The graphic of equation 27 could be seen in the corresponding notebook. In this case the corresponding Fourier coefficients are c_n:

[2.6660,0.3817,0.5284,0.1772,0.2075,0.0584, 0.0705,0.0,0.0116,-0.0219,-0.0096......]

where:

$$c_n = \frac{2}{\pi}\int_0^{\pi} dt x(t)\sin(nt) \tag{29}$$

we see that the convergency is rather slow, using the relation for the velocity implied in the Fourier expansion:

$$v_n = \sum_1^{\infty} nc_n \tag{30}$$

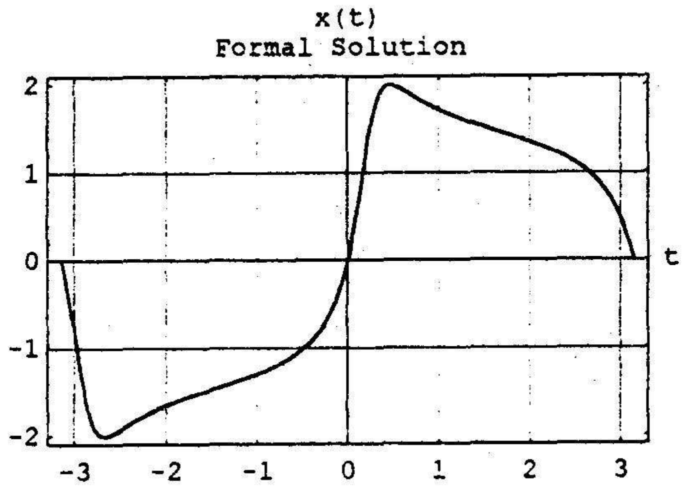

Fig. 7 Solution to Smith equation

since the sum of the first 11 terms give 7.4907 instead of 4.4721 and the first negative term appears for n=9.

6.2. Delamotte method to Smith equations

Applying the external force equation I.3 to Smith equation 21 we obtain:

$$\delta x'' = -x'_{ans} f(x_{ans} + \delta x) - g(x_{ans} + \delta x) + \omega^2 x_{ans} - F_{ans} \tag{31}$$

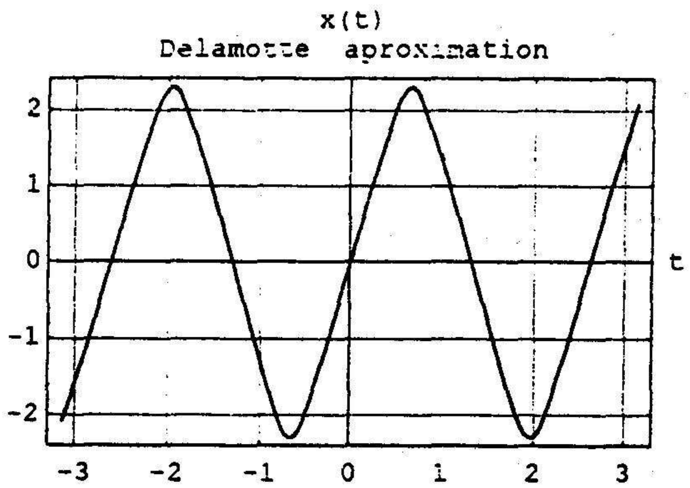

Fig. 8 Order 4 Delamotte approximation to Smith equation

which gives the correct answer for the Van der Pol equation and in our case:

$$-4x'_{ans}(x^2_{ans}-1)+(\omega^2-1)x_{ans}-x_{ans}(x^2_{ans}-2)^2-F_{ans}\sim 0 \tag{32}$$

The class of systems to which we have consider the Delamotte method is called dissipative systems. A system of differential equations is dissipative if whatever the initial condition, there exist some t_0 that for $t > t_0$, x(t) the solution is contained in a bounded subset. The characteristic of dissipative systems with a unique cycle limit is that for a special set of initial conditions there is a periodic orbit, this has as a consequence that the numerical solution could become instable, on the other hand from equation I.7 and the fact that Delamotte expansion gives the limit cycle, we can choose the $\underline{a}$, (Delamotte, 1993) we certainly see that in this case the Delamotte expansion could not be the Fourier expansion, which is unique. The problem is, what happens when your choice of $\underline{a}$ is a number different from the Fourier coefficient, does the expansion converge, gives the right velocity? Does the choice of $\underline{a}$ equal to the Fourier coefficient generate a super convergent sequence?

For a dissipative differential equation system $x' = f(x)$ with a unique limit cycle, there is one Delamotte approximation which is the Fourier expansion of the periodic orbit. All Delamotte solutions are structurally stable in relation to the limit cycle.